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Entropy of Pairs of Dual Attractors in Six and Seven Dimensions

El Hassan Saidi*

Lab/UFR- Physique des Hautes Energies, Faculté des Sciences, Rabat, Morocco,
E-mail: `h-saidi@fsr.ac.ma`

Antonio Segui

Departamento de Fisica Teorica, Universidad de Zaragoza, 50009-Zaragoza, Spain,
E-mail: `segui@unizar.es`

ABSTRACT: We study the attractor mechanism of dual pairs of black brane bounds in $\mathcal{N}=2$ supergravity in six and seven dimensions. First, we consider the effective potentials of the *6D* and *7D* black branes as well as their entropies. The contribution coming from the $SO(1,1)$ factor of the moduli spaces M_{6D} and M_{7D} of these theories is carefully analyzed and it is used to motivate the study of the dual black branes bounds; which in turn allow to fix the critical value of the dilaton at horizon. The attractor eqs of the black branes and the bound pairs are derived by combining the criticality conditions of the corresponding effective potentials and the Lagrange multiplier method capturing constraints eqs on the fields moduli.

KEYWORDS: *6D/7D* black branes, attractor mechanism, electric/magnetic duality, BPS and non BPS states, entropy.

*College SPC, Academie Hassan II des Sciences et Techniques, Rabat, Morocco

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1. Introduction

Supersymmetric and non supersymmetric black attractors have received an increasing interest in the framework of supergravity theories [1]-[8]; especially in the case of those supergravity models embedded in *10D* superstrings and *11D* M-theory compactifications [9]-[27]. New solutions to the attractor equations describing BPS and non-BPS states have been obtained and many results concerning supergravity theories in *four* and *higher* dimensional space times have been derived [28]-[40]. For reviews, see for instance [41]-[46]. In this paper, we contribute to this matter by studying the attractor mechanism and the entropy $\mathcal{S}$ of the two following *6D/7D* black brane systems:

1) the first system we consider concerns generic *Electrically* charged *Black Branes* (*EBB* for short) in $\mathcal{N}=2$ supergravity theory in six and seven dimensional space times. If most of the basic properties of these *EBB*s with electric charges

$$q_\Lambda \quad \neq \quad 0 \quad , \quad \Lambda = 1, \ldots \quad , \tag{1.1}$$

but no magnetic charges

$$g_\Lambda \quad = \quad 0 \quad , \quad \Lambda = 1, \ldots \quad , \tag{1.2}$$

are quite known; there are nevertheless some specific properties that need more studies. Here, we would like to shed more light on the *EBB* entropy and the electric/magnetic duality which, as we will show, turn out to be strongly related:
a) Concerning the entropy $S_{EBB} = S_{EBB}(q)$ of the *EBB* black attractors in *6D* and *7D*, it turns out that it takes a very remarkable value[1] namely,

$$S_{EBB} \quad = \quad 0 \quad . \tag{1.3}$$

This degenerate value will be analyzed in details throughout this study by using the criticality method; but to fix the ideas think about it as given by the $g_\Lambda \to 0$ limit of the following relation to be explicitly derived in this work,

$$S_{EBB} = \frac{1}{2} \lim_{g_\Lambda \to 0} \left(\sqrt{|q^2 g^2|} \right) = 0, \tag{1.4}$$

with $q^2 = \sum_\Lambda \left(q_\Lambda q^\Lambda \right)$ and $g^2 = \sum_\Lambda \left(g_\Lambda g^\Lambda \right)$.
In an attempt to analyze what kind of information we can extract from the *classical* relation $S_{EBB}(q) = 0$, we ended with the conclusion that this degenerate value could be interpreted as *a singular limit* of a bound state of the following pair of dual black attractor

$$EBB\text{-}MBB \quad , \tag{1.5}$$

[1] Black holes could be either small or large depending on whether the corresponding classical horizon area is zero or non zero [47, 48, 49, 12]. If we naively apply the Bekenstein-Hawking entropy-area formula to the small black holes, their entropy vanishes and the expected quantum degrees of freedom seem to totally *disappear*. This discrepancy comes from the fact that the general relativity is only a classical effective theory of quantum gravity opening then a way to deal with small black holes in connection with R^2 corrections and supersymmetry enhancement in near horizon geometry [50, 48, 49, 12, 51]. Small black holes exist also in higher dimensions. In $5D$, an explicit study of small black holes in $\mathcal{N}=2$ and $\mathcal{N}=4$ supergravity can be found in [52] and refs therein.

where *MBB* stands for the magnetic dual of *EBB*.
The black attractor bound state *EBB-MBB* will be introduced and commented succinctly below; see the point (**2**) of this motivating presentation. But explicit details and extensive comments will be given in the *section 6* of this work.
b) Concerning the electric/magnetic duality, it is used to deal with the *Magnetically* charged *Black Branes* (*MBB*). Roughly, this duality exchanges the charges of the *EBB* and the corresponding *MBB* dual along the standard correspondence,

$$EBB \quad \xleftrightarrow{\text{electric/magnetic duality}} \quad MBB \quad . \tag{1.6}$$

In this study, we will show that electric/magnetic duality is in fact a *universal* symmetry of *EBB* and *MBB* attractors. It exchanges not only the electric $\{q_\Lambda\}$ and magnetic $\{g_\Lambda\}$ charges ($q_\Lambda \leftrightarrow g_\Lambda$); but also the effective scalar potentials $\mathcal{V}_{EBB}$ and $\mathcal{V}_{MBB}$ as well as the corresponding entropies $\mathcal{S}_{EBB}$ and $\mathcal{S}_{MBB}$ as illustrated below,

$$\begin{array}{lcl} \underline{EBB} & \xleftrightarrow{\text{electric/magnetic duality}} & \underline{MBB} \\ q_\Lambda & \longleftrightarrow & g_\Lambda \\ \mathcal{V}_{EBB} & \longleftrightarrow & \mathcal{V}_{MBB} \\ \mathcal{S}_{EBB} & \longleftrightarrow & \mathcal{S}_{MBB} \quad . \end{array}$$

From this correspondence, we immediately conclude that the entropy $\mathcal{S}_{MBB} = \mathcal{S}_{MBB}(g)$ of the magnetically charged black brane *MBB* should be identically zero; in agreement with eq(1.4). The relation $\mathcal{S}_{MBB} = 0$ will be rigourously derived in *sub-section 5.2.*
We also learn that the scalar potentials $\mathcal{V}_{EBB}$ and $\mathcal{V}_{MBB}$ are intimately related as it will be explicitly shown in *section 5.*

2) the second system that we want to study in this paper concerns precisely generic *Dual Black Brane Pairs* (dual pairs *DP* for short).
A *DP* attractor can defined as a bound state consisting of an electrically charged brane *EBB* and its magnetic dual *MBB*. Formally, we can represent a generic *DP* bound state either as in eq(1.5) or roughly, by using group theory representation language, like a doublet

$$DP \quad \sim \quad \begin{pmatrix} EBB \\ MBB \end{pmatrix} \quad . \tag{1.7}$$

In this set up, the *EBB* attractor considered in point (**1**), with the degenerate entropy $S_{EBB} = 0$, can be thought of as corresponding to the *singular* limit

$$DP \quad \xrightarrow{g_\Lambda \to 0} \quad \begin{pmatrix} EBB \\ 0 \end{pmatrix} \quad , \tag{1.8}$$

describing a singular geometry where the horizon area A_{MBB} of the *MBB* attractor shrinks to a singular point ($A_{MBB} \to 0$).
The same picture is valid for the dual *MBB* attractor which corresponds to the degenerate limit,

$$DP \quad \xrightarrow{q_\Lambda \to 0} \quad \begin{pmatrix} 0 \\ MBB \end{pmatrix} \quad , \tag{1.9}$$

describing the electric/magnetic dual of eq(1.8).
Before proceeding ahead, we would like to notice that the results we will derive below for the *DP* attractors apply as well to:
i) the dyonic *Black String* (*BS* for short) of the *6D* $\mathcal{N}=2$ supergravity,
ii) the *DP* brane bounds in all supergravity theories with scalar manifolds of the form $SO(1,1)\times(G/H)$.
Regarding the *6D* black string *BS*, it can be viewed as a particular representation of the electric magnetic duality group. The *BS* is a pure singlet while *DP* is based on the *pair* (1.7).
Moreover, it is interesting to have in mind that, despite their geometric differences, the *BS* and *DP* entropy formulas are also comparable. This feature can be explicitly checked by comparing the $\mathcal{S}_{DP}$ formula (1.4) and the *BS* entropy relation $\mathcal{S}_{BS}$ to be derived in *section 4* eq(4.13), and which we recall below,

$$\mathcal{S}_{BS}=\frac{|q_0g_0|}{2}=\frac{1}{2}\sqrt{q_0^2g_0^2}\ , \tag{1.10}$$

with q_0 and g_0 being respectively the electric and magnetic charges of the *6D* black string. As we see, the above $\mathcal{S}_{BS}$ expression and the $\mathcal{S}_{DP}$ relation (1.4) have more a less the same charge dependence structure.
Concerning the second feature; it has been pointed out in [46, 55, 56], that supergravity theories with scalar manifolds having an $SO(1,1)$ factor would have zero entropies. The result $\mathcal{S}_{\text{EBB}}=0$ of eq(1.3) and, up on using electric/magnetic duality

$$\mathcal{S}_{\text{MBB}}=0\ , \tag{1.11}$$

should be then thought of as special relations that are valid as well for supergravity theories beyond those embedded in *10D* type IIA superstring and *11D* M-theory on K3 we are considering in this work.

On the other hand, we will also take the opportunity of the use of the criticality condition of the black branes effective potentials to develop a tricky approach to get the BPS and non BPS states solutions by using an adapted Lagrange multiplier method. Details on this issue will be given in *section 5* of this study. BPS and non BPS black holes as well as black membranes are distinguished by the values (5.26-5.28,5.45) of the Lagrange multipliers at the minimum of the effective potential. The Lagrange multipliers $\{\lambda^{\Lambda\Sigma}\}$ given by eq(4.35) capture the constraint eqs(5.8,5.14) on the fields coordinates $\{L_{\Lambda\Sigma}\}$ (4.36) that are used to parameterize the moduli space of the theory.

The organization of this paper is as follows: In section 2, we describe briefly some useful tools; in particular the derivation of the singular value (1.3). In section 3, we study the *EBB* and *MBB* attractors as well as the *DP* black attractor bounds in *6D* and *7D* $\mathcal{N}=2$ supergravity theories. In section 4, we consider with details the *6D* black string. We show, amongst others, that its entropy is invariant under electric/magnetic duality and conclude with a general result on dyonic duals pairs of black branes. In section 5, we study the BPS and non BPS black attractors in *6D* by using a new method. This

approach is based on combining the criticality of the effective potential and the Lagrange multiplier method capturing constraint eqs on the field moduli. Using this approach we derive the attractors eqs of the *6D* black hole *BH* and *6D* black membrane *BM*. We also give the explicit solutions as well as their entropies. In section 6, we derive the effective potential $\mathcal{V}_{DP}$ of the dyonic dual pair bound $DP \equiv BH\text{-}BM$. Then we study the attractor mechanism for the dyonic *DP* and derive the general formula for its entropy $\mathcal{S}_{DP}$. This result obtains for the 6D apply as well to the dyonic black hole-black 3- brane (*BH-B3B*) and the (*BS-BM*) bound state of the 7D theory. In section 7, we give the conclusion and discussion and in section 8, we give an appendix.

2. General tools

To exhibit explicitly the particular features

$$\begin{array}{lllllll} \mathcal{S}_{\text{EBB}}^{6D} & = & 0 & , & \mathcal{S}_{\text{EBB}}^{7D} & = & 0 \quad , \\ \mathcal{S}_{\text{MBB}}^{6D} & = & 0 & , & \mathcal{S}_{\text{MBB}}^{7D} & = & 0 \quad , \end{array} \tag{2.1}$$

of the entropies of the electrically charged *EBB* and the magnetically charged *MBB* in six and seven space time dimensions, it is interesting to start by describing briefly some useful results.

We begin by recalling that the moduli space $\boldsymbol{M}_{6D}^{N=2}$ of the *6D* $\mathcal{N}=2$ supergravity theory embedded in *10D* type IIA superstring on K3 is given by the following Lie group coset

$$\begin{array}{lll} \boldsymbol{M}_{6D}^{N=2} & = & SO\left(1,1\right) \times G_6 \quad , \\ G_6 & = & \frac{SO(4,20)}{SO(4)\times SO(20)} \quad . \end{array} \tag{2.2}$$

A quite similar factorization holds for the scalar manifold $\boldsymbol{M}_{7D}^{N=2}$ of the $\mathcal{N}=2$ supergravity theory in *7D* space time embedded in *11D* M-theory on K3. It reads as follows

$$\begin{array}{lll} \boldsymbol{M}_{7} & = & SO\left(1,1\right) \times G_7 \quad , \\ G_7 & = & \frac{SO(3,19)}{SO(3)\times SO(19)} \quad . \end{array} \tag{2.3}$$

As we see, the two scalar manifolds $\boldsymbol{M}_{6D}^{N=2}$ and $\boldsymbol{M}_{7D}^{N=2}$ are given by the product of two factors namely $SO\left(1,1\right)$ and G_n with $n=6,7$.

The real one dimensional factor $SO\left(1,1\right)$ is parameterized by a real field variable σ, to be interpreted as the *6D* (resp. *7D*) dilaton $\sigma=\sigma\left(x\right)$.

The factor G_6 is real *80* dimensional manifold parameterized by the real field coordinates,

$$\phi^{aI}\left(x\right) \quad \simeq \quad \left(\underline{\mathbf{4}},\underline{\mathbf{20}}\right) \quad , \tag{2.4}$$

transforming in the bi-fundamental of the $SO\left(4\right)\times SO\left(20\right)$ isotropy symmetry with $a=1,...,4$ and $I=1,...,20$.

The factor G_7 is real *57* dimensional manifold parameterized by the field coordinates

$$\xi^{\alpha i}\left(x\right) \quad \simeq \quad \left(\underline{\mathbf{3}},\underline{\mathbf{19}}\right) \quad , \tag{2.5}$$

transforming in the bi-fundamental of the $SO\left(3\right)\times SO\left(19\right)$ isotropy symmetry with $\alpha=1,2,3$ and $i=1,...,19$.

As the technical analysis of eqs(2.2) and (2.3) is quite similar, we will focus our attention mainly on the *6D* theory and just give the results for the *7D* case.

2.1 Effective potential

The effective potential $\mathcal{V}$ of black attractors in generic space time D- dimensional extended supergravity, including *6D* $\mathcal{N}=2$, have been studied in [46]; see also [55] as well as the appendix of this paper. The general form of this potential reads formally, in terms of the geometric $\mathcal{Z}_{geo}$ and the matter $\mathcal{Z}_{matter}$ central charges, as follows

$$\mathcal{V}(\phi) \sim |\mathcal{Z}_{geo}(\phi)|^2 + |\mathcal{Z}_{matter}(\phi)|^2 .$$

Notice that $\mathcal{Z}_{geo}$ has contributions coming from the physical charges of the various gauge fields of the gravity supermultiplet while $\mathcal{Z}_{matter}$ has contributions coming from the gauge fields in the matter sector.
In the case of 6D $\mathcal{N}=2$ non chiral supergravity, we have the following gauge field strengths,

$$\begin{array}{lclcl} \text{gravity multiplet} & : & \mathcal{H}_3 = d\mathcal{B}_2 & , & \mathcal{F}_2^a = d\mathcal{A}_1^a \quad , \\ \text{matter multiplets} & : & \mathcal{F}_2^I = d\mathcal{A}_1^I & & \quad , \end{array} \tag{2.6}$$

together with their magnetic duals $\mathcal{G}_3$, $\mathcal{G}_4^a$ and $\mathcal{G}_4^I$. So $\mathcal{Z}_{geo}^{6D,N=2}$ and $\mathcal{Z}_{matter}^{6D,N=2}$ have contributions from the charges of $(\mathcal{H}_3, \mathcal{G}_3)$, $\left(\mathcal{F}_2^a, \mathcal{F}_2^I\right)$ and $\left(\mathcal{G}_4^a, \mathcal{G}_4^I\right)$; and then the full effective potential $\mathcal{V}^{6D,N=2}$ involves three blocks namely $\mathcal{V}_{black\ string}$, $\mathcal{V}_{black\ hole}$ and $\mathcal{V}_{black\ membrane}$. Notice also that in eq(2.6), $\mathcal{B}_2 = \frac{1}{2}dx^\mu dx^\nu B_{[\mu\nu]}$ is the usual NS-NS $B_{\mu\nu}$- *field* in 6D, the gauge fields $\mathcal{A}_\mu^a$ stand for the four graviphotons and $\mathcal{A}_\mu^I$ for the twenty Maxwell fields of the non chiral 6D supergravity embedded in type IIA superstring on K3, see eqs(3.29-3.30) to fix the ideas.
Following [46] and [55], we can compute explicitly the various contributions $\mathcal{V}_{black\ string}$, $\mathcal{V}_{black\ hole}$ and $\mathcal{V}_{black\ membrane}$ by using the following generic relations,

$$\begin{array}{lclclc} \mathcal{V}_{black\ string} & \sim & \left|\mathcal{Z}_{geo}^{BS}\right|^2 & + & \left|\mathcal{Z}_{matter}^{BS}\right|^2 & , \\ \mathcal{V}_{black\ hole} & \sim & \left|\mathcal{Z}_{geo}^{BH}\right|^2 & + & \left|\mathcal{Z}_{matter}^{BH}\right|^2 & , \\ \mathcal{V}_{black\ membrane} & \sim & \left|\mathcal{Z}_{geo}^{BM}\right|^2 & + & \left|\mathcal{Z}_{matter}^{BM}\right|^2 & . \end{array}$$

These contributions, which are respectively associated with $(\mathcal{H}_3, \mathcal{G}_3)$, $\left(\mathcal{F}_2^a, \mathcal{F}_2^I\right)$ and $\left(\mathcal{G}_4^a, \mathcal{G}_4^I\right)$, will be studied later on; they are given by eqs(4.2), (5.2), (5.37). With these relations in mind, we turn now to study some specific properties of these potentials.
One of the consequences of the factorization (2.2) of the manifold $\boldsymbol{M}_{6D}^{N=2}$ is that the *EBB* (resp. *MBB*) effective scalar potential

$$\mathcal{V}_{6D}^{\text{SBB}} = \mathcal{V}_{6D}^{\text{SBB}}(\sigma, \phi) \quad , \tag{2.7}$$

where the upper index *SBB* stands either for *EBB* or *MBB*, factorizes as well like

$$\mathcal{V}_{6D}^{\text{SBB}} = \mathcal{V}_{SO(1,1)} \times \mathcal{V}_{G_6} \quad . \tag{2.8}$$

The term in the right hand of the above relation,

$$\mathcal{V}_{SO(1,1)} = \mathcal{V}_{dil}(\sigma) \quad , \tag{2.9}$$

is the dilaton contribution to eq(2.8); it has no dependence in the local field coordinates ϕ^{aI}; that is no dependence in the matter fields of the Maxwell sector of the theory,

$$\frac{\partial \mathcal{V}_{SO(1,1)}}{\partial \phi^{aI}} = 0. \tag{2.10}$$

We will see later on that this contribution is given by the typical remarkable relation

$$\mathcal{V}_{SO(1,1)}(\sigma) \quad \simeq \quad \exp(\mathrm{n}\sigma) \quad , \tag{2.11}$$

where the number n depends on the type of the black brane we are dealing with. More precisely, we have the following values [55, 56],

$$\begin{array}{lcccll} \text{6D black string} & : & \mathrm{n} & = & \pm 4 & , \\ \text{6D black hole} & : & \mathrm{n} & = & -2 & , \\ \text{6D black membrane} & : & \mathrm{n} & = & +2 & . \end{array} \tag{2.12}$$

The factor $\mathcal{V}_{G_6}$ of (2.8) has no dependence in the dilaton

$$\begin{array}{lcll} \mathcal{V}_{G_6} & = & \mathcal{V}_{G_6}(\phi) & , \\ \frac{\partial \mathcal{V}_{G_6}}{\partial \sigma} & = & 0 & , \end{array} \tag{2.13}$$

it describes the contribution of the matter fields $\{\phi^{aI}\}$ in the Maxwell sector of the 6D $\mathcal{N}=2$ supergravity theory. The explicit field expression of $\mathcal{V}_{G_6}$ in terms of the ϕ^{aI} will be given later on.

2.2 Criticality condition

First we study the electrically (*resp.* magnetically) charged black brane *EBB* (*resp. MBB*). Then we consider the case of the dyonic black string *BS*.
The critical values $(\sigma, \phi) = (\sigma_c, \phi_c)$ of the effective scalar potential $\mathcal{V}_{6D}^{\mathrm{SBB}}$ (2.7-2.8) are obtained by solving the constraint equations

$$\begin{array}{lcll} \frac{\partial \mathcal{V}_{6D}^{\mathrm{SBB}}}{\partial \sigma} & = & 0 & , \\ \frac{\partial \mathcal{V}_{6D}^{\mathrm{SBB}}}{\partial \phi^{aI}} & = & 0 & , \end{array} \tag{2.14}$$

which, due to the factorization property (2.8), simplify like,

$$\begin{array}{lcll} \frac{\partial \mathcal{V}_{SO(1,1)}}{\partial \sigma} & = & 0 & , \\ \frac{\partial \mathcal{V}_{G_6}}{\partial \phi^{aI}} & = & 0 & . \end{array} \tag{2.15}$$

The critical value σ_c of the dilaton that extremize the potential $\mathcal{V}_{SO(1,1)}$, and then $\mathcal{V}_{6D}^{\mathrm{SBB}}$, is obtained by computing

$$\frac{\partial \mathcal{V}_{SO(1,1)}}{\partial \sigma} \simeq \frac{\partial \left[e^{n\sigma}\right]}{\partial \sigma} = n e^{n\sigma} = 0 \ , \tag{2.16}$$

from which we learn that the critical point corresponds to the following infinite value,

$$n\sigma_c \longrightarrow -\infty. \tag{2.17}$$

For $n > 0$, $\sigma_c \longrightarrow -\infty$ and for $n < 0$, $\sigma_c \longrightarrow +\infty$.
Putting this value back into $\mathcal{V}_{SO(1,1)}$ eq(2.11), we see that the value of the potential $\mathcal{V}_{SO(1,1)}$ at the critical point vanishes identically; i.e,

$$\left[\mathcal{V}_{SO(1,1)}\right]_{\sigma=\sigma_c} = 0 \quad . \tag{2.18}$$

Because of the factorization (2.8), we also have

$$\left[\mathcal{V}_{6D}^{\mathrm{SBB}}\right]_{\sigma=\sigma_c} = 0 \quad , \tag{2.19}$$

leading as well to the zero entropy relation,

$$\mathcal{S}_{6D}^{\mathrm{SBB}} = 0 \quad , \tag{2.20}$$

in agreement with eq(1.3).
For the *dyonic* 6D black string, the situation is different. The form of the corresponding effective potential $\mathcal{V}_{BS}$ has the following field moduli factorization,

$$\mathcal{V}_{BS} = \mathcal{V}_{SO(1,1)}\left(\sigma\right) \times \mathcal{V}_6\left(\phi\right) + \mathcal{V}_{SO(1,1)}\left(-\sigma\right) \times \tilde{\mathcal{V}}_6\left(\phi\right) \quad , \tag{2.21}$$

where $\mathcal{V}_6\left(\phi\right)$ stands for the contribution coming from the electric charge and $\tilde{\mathcal{V}}_6\left(\phi\right)$ the contribution coming from the magnetic charge.
As we will see in details later, it turns out that the solving of the criticality condition of $\mathcal{V}_{BS}$ leads to a finite critical value of the dilaton

$$|\sigma_c| < \infty \ . \tag{2.22}$$

Substituting this value back into $\mathcal{V}_{BS}$, we obtain a positive definite value of the effective potential at the minimum,

$$\left[\mathcal{V}_{BS}\left(\sigma\right)\right]_{\sigma=\sigma_c} > 0 \ , \tag{2.23}$$

leading in turn to a on a zero value of the entropy $\mathcal{S}_{BS}$ for the dyonic 6D black string. The value of $\mathcal{S}_{BS}$ is given by eq(1.10); it will be computed explicitly later on, see eq(4.13).

3. Duality symmetry and entropy

First, we describe some useful aspects on:
(**1**) the *6D* and *7D* attractors and the dual pairs,
(**2**) the gauge invariant n-forms in generic d- dimensions; in particular the electric/magnetic duality [57, 58, 59] and the fluxes used to define the various electric and magnetic charges.
Then, we study the *"dyonic"* attractors in *6D* and *7D*. We will distinguish the two following cases:
(**a**) the *6D* Black String *BS*; behaving as a singlet under electric/magnetic duality

$$\left(BS\right) \ . \tag{3.1}$$

No analogous object exists in *7D*.
(**b**) Bound states of dual *EBB* and *MBB* behaving as pairs under electric magnetic duality

$$\begin{pmatrix} EBB \\ MBB \end{pmatrix} . \tag{3.2}$$

The possible candidates for these bound pairs are:
(**i**) the *6D* Black Hole - Black Membrane (*BH-BM*),
(**ii**) the *7D* Black Hole - Black 3- Brane (*BH-B3B*),
(**iii**) the *7D* Black String - Black Membrane (*BS-BM*).
Below, we shall focus our attention in a first step on the special *6D* dyonic string *BS* and its entropy $\mathcal{S}_{BS}$.
Then, we study the basic properties of the *BH* and *BM* black attractors separately. This study can be viewed as a prelude to *BH-BM* bound.
More details on the dual pair *BH-BM* in six dimensions and its analogues in *7D* will be considered in section 6 and the discussion section.

3.1 6D and 7D black attractors

Electric/magnetic duality permutes electrically charged objects and their magnetic charged duals. In *10D* type II superstrings and *11D* M-theory compactifications down to *d- dimensions*, this discrete symmetry relates those pairs of p_1- and p_2- dimensional black objects with the condition

$$p_1 + p_2 = d - 4 \qquad , \qquad d \geq 4 \ . \tag{3.3}$$

From this relation, one recognizes:
(**1**) the *4D* dyonic black hole corresponding to $p_1 + p_2 = 0$.
(**2**) the *6D* dyonic black string corresponding to $p_1 + p_2 = 2$.
(**3**) the *8D* dyonic black membrane corresponding to $p_1 + p_2 = 4$.
In six and seven dimensions we are interested in we have the following:

6D case

In the non chiral 6D $\mathcal{N} = 2$ supergravity theory embedded in 10D type IIA superstring on K3, the relation (3.3) reads as,

$$p_1 + p_2 = 2 \ , \tag{3.4}$$

and can be solved in three ways like:
(**a**) the case $(p_1, p_2) = (1, 1)$ describing a dyonic black string (*BS*).
The 6D *BS* attractor carries both an electric charge q_0 and a magnetic charge g_0 associated with the gauge invariant 3-form field strength

$$\mathcal{H}_3 = d\mathcal{B}_2 \ , \tag{3.5}$$

of the $\mathcal{N} = 2$ supergravity multiplet.
(**b**) the case $(p_1, p_2) = (0, 2)$ describing a magnetic black hole (*BH*).

In *6D*, the *BH* attractor carries *24* magnetic charges $\mathrm{g}_{\Lambda}^{BH}$ (g_{Λ} for short) associated with the gauge invariant fields strengths

$$\mathcal{F}_2^{\Lambda} = d\mathcal{A}_1^{\Lambda} , \tag{3.6}$$

of the $\mathcal{N} = 2$ supergravity theory. The *6D BH* hole has no electric charge,

$$\mathrm{q}_{\Lambda}^{BH} = 0. \tag{3.7}$$

(**c**) the case $(p_1, p_2) = (2, 0)$ describing an electric *6D* black membrane (BM) carrying *24* electric charges $\mathrm{q}_{\Lambda}^{BM}$ (q_{Λ} for short) related to $\mathrm{g}_{\Lambda}^{BH}$ under electric magnetic duality.
The *6D* black membrane has no magnetic charge

$$\mathrm{g}_{\Lambda}^{BM} = 0. \tag{3.8}$$

The above BH and the BM attractors are related by electric/magnetic duality in six dimensions. As such, the bound state made of the *6D* black hole BH and the *6D* black membrane BM

$$6D \quad : \quad BH\text{-}BM \quad \equiv \quad \begin{pmatrix} BH \\ BM \end{pmatrix} \quad , \tag{3.9}$$

form a *dyonic pair* of black attractors with 24 electric and 24 magnetic charges

$$\{\mathrm{q}_{\Lambda}, \mathrm{g}_{\Lambda}\} \qquad , \qquad \Lambda = 1, ..., 24. \tag{3.10}$$

Viewed as a single entity, the composite state *BH-BM* should, à priori, share the basic features of the dyonic black string BS; in particular sharing aspects of the effective potentials and their entropies. We will study these features details later on.

7D case

In the case of *7D* $\mathcal{N} = 2$ supergravity theory embedded in *11D* M-theory on K3, the relation (3.3) becomes

$$p_1 + p_2 = 3 \tag{3.11}$$

and it is solved in four manners as follows:
(**a**) the case $(p_1, p_2) = (0, 3)$ describing a magnetic *7D* black hole (BH),
(**b**) the case $(p_1, p_2) = (3, 0)$ describing an electric *7D* black 3-brane ($B3B$), dual to the black hole.
(**c**) the case $(p_1, p_2) = (1, 2)$ describing a magnetic *7D* black string (BS).
(**d**) the case $(p_1, p_2) = (2, 1)$ describing an electric *7D* black 2-brane (BM), dual to the black string.
The 7D $\mathcal{N} = 2$ supergravity theory embedded in *11D* M-theory on K3 has the following abelian gauge symmetry,

$$U_{NS}(1) \times U^3(1) \times U^{19}(1) . \tag{3.12}$$

The 7D black hole BH and black 3-brane $B3B$ are charged under the $U^{22}(1)$ gauge symmetry of the supergravity theory while the 7D black string BS and black membrane BM are charged under the gauge invariant 3-form and its dual 4-form.

Notice that in *7D* $\mathcal{N}=2$ supergravity theory, we have no dyonic singlet; but rather two kinds of dyonic pairs:
(i) the pair

$$BH\text{-}B3B \quad \equiv \quad \begin{pmatrix} BH \\ B3B \end{pmatrix} \quad , \tag{3.13}$$

carrying *22* electric charges $\{q_1, ..., q_{22}\}$ and *22* magnetic ones $\{g_1, ..., g_{22}\}$.
(ii) the pair

$$BS\text{-}BM \quad \equiv \quad \begin{pmatrix} BS \\ BM \end{pmatrix} \quad , \tag{3.14}$$

carrying an electric charge q_0 and a magnetic charge g_0.
Notice also that there is a correspondence between the attractors in 6D and 7D space time dimensions. We have,

$$\begin{array}{lcll} 6D : BS & \longleftrightarrow & 7D : BS\text{-}BM & , \\ 6D : BH\text{-}BM & \longleftrightarrow & 7D : BH\text{-}B3B & . \end{array} \tag{3.15}$$

This property is a consequence following from the relation between *11D* M- theory and *10D* type IIA superstring; which after compactification on K3, descends to the *6D* and the *7D* space times.

3.2 Dyonic attractors in 6D supergravity

To start recall that in the d- dimensional space time, a gauge invariant $(p+2)$- form field strength $(p \leq d-2)$,

$$\mathcal{F}_{p+2} \quad = \quad \tfrac{1}{(p+2)!} dx^{\mu_{p+2}} \ldots dx^{\mu_1} \mathcal{F}_{\mu_1 \cdots \mu_{p+2}} \quad , \tag{3.16}$$

with a $(p+1)$- form gauge connection

$$\mathcal{A}_{p+1} \quad = \quad \tfrac{1}{(p+1)!} dx^{\mu_{p+1}} \ldots dx^{\mu_1} \mathcal{A}_{\mu_1 \cdots \mu_{p+1}} \quad , \tag{3.17}$$

has a Poincaré (magnetic) dual given by

$$\mathcal{G}_{d-p-2} \quad = \quad {}^{\star}\mathcal{F}_{p+2} \quad . \tag{3.18}$$

with the usual property

$${}^{*}\mathcal{G}_{d-p-2} = -\left(-\right)^{(p+2)(d-p-2)} \mathcal{F}_{p+2}. \tag{3.19}$$

Expanding $\mathcal{G}_{d-p-2}$,

$$\mathcal{G}_{d-p-2} \quad = \quad \tfrac{1}{(p+2)!(d-p-2)!} dx^{\mu_D} \cdots dx^{\mu_{p+3}} \mathcal{G}_{\mu_{p+3} \cdots \mu_d} \quad , \tag{3.20}$$

we also have

$$\mathcal{G}^{\mu_{p+3} \cdots \mu_d} \quad = \quad \mathcal{F}_{\mu_1 \cdots \mu_{p+2}} \varepsilon^{\mu_1 \cdots \mu_{p+2} \mu_{p+3} \cdots \mu_d} \quad , \tag{3.21}$$

with $\varepsilon^{\mu_1 \cdots \mu_{p+2} \mu_{p+3} \cdots \mu_d}$ being the d- dimensional completely antisymmetric tensor.
The magnetic charge g (resp electric charge q) associated with the field strength $\mathcal{F}_{p+2}$ (resp. $\mathcal{G}_{d-p-2}$) can be defined as

$$\begin{array}{ll} \mathrm{g} = \int_{S^{p+2}} \mathcal{F}_{p+2} & , \\ \mathrm{q} = \int_{S^{d-p-2}} \mathcal{G}_{d-p-2} & . \end{array} \tag{3.22}$$

Using the normalized n- volume form Ω_n of the real n- sphere $\mathbb{S}^n$,

$$\int_{\mathbb{S}^n} \Omega_n = 1 \ , \qquad n = p+2 \quad \text{or} \quad d-p-2 \ , \tag{3.23}$$

we can also express the gauge invariant field strengths as follows,

$$\begin{array}{lcll} \mathcal{F}_{p+2} & = & \mathrm{g}\ \Omega_{p+2} & , \\ \mathcal{G}_{d-p-2} & = & \mathrm{q}\ \Omega_{d-p-2} & , \end{array} \tag{3.24}$$

with

$$\Omega_{p+2} \wedge \Omega_{d-p-2} \quad \simeq \quad V_d \quad . \tag{3.25}$$

where V_d is the volume real d- form of the space time. We also have

$$\mathcal{F}_{p+2} \wedge \mathcal{G}_{d-p-2} \quad \simeq \quad \mathrm{gq} V_d \quad , \tag{3.26}$$

with the following quantization condition relating electric and magnetic sectors,

$$\mathrm{gq} \quad = \quad 2\pi k \quad , \quad k \ \ \text{integer} \quad . \tag{3.27}$$

Seen that the analysis for *6D* and the analysis for *7D* are quite similar, we shall fix our attention in what follows on the *6D* $\mathcal{N} = 2$ non chiral supergravity theory; too particularly on the case of *6D* supergravity models embedded in *10D* type IIA superstring on K3. There, the field theory spectrum following from the compactification of *10D* type IIA superstring on K3, involves the two supersymmetric multiplets namely the gravity supermultiplet and the Maxwell supermultiplets:

(**1**) *the gravity supermultiplet*
This supermultiplet contains, in addition to fermions, the following bosonic fields:

(**i**)	the 6D gravity field	:	$g_{\mu\nu} = g_{\mu\nu}(x)$	, $\mu, \nu = 0, ..., 5$,
(**ii**)	the NS-NS B- field	:	$B_{\mu\nu} = B_{\mu\nu}(x)$	,
(**iii**)	four $U(1)$ gauge fields	:	$\mathcal{A}_\mu^a = \mathcal{A}_\mu^a(x)$	, $a = 1, ..., 4$,
(**iv**)	the 6D dilaton	:	$\sigma = \sigma(x)$	.

(**2**) *the Maxwell gauge sector*
This sector involves twenty Maxwell supermultiplets with the following bosons:

(**i**)	twenty abelian gauge fields	:	$\mathcal{A}_\mu^I = \mathcal{A}_\mu^I(x)$	, $I = 1, ..., 20$,
(**ii**)	twenty quartets of scalars	:	$\phi_{aI} = \phi_{aI}(x)$	.

The abelian gauge symmetry group of the *6D* $\mathcal{N}=2$ supergravity theory that we are considering here can be cast as follows

$$U_{NS}(1) \times U^4(1) \times U^{20}(1)\,. \tag{3.28}$$

The $U_{NS}(1)$ factor is the abelian gauge symmetry associated with the NS-NS gauge field $B_{\mu\nu}$ and field strength $\mathcal{F}_3$ that we have denoted earlier as $\mathcal{H}_3$.
The abelian factor $U^4(1)$ is the gauge symmetry with the four gauge fields $\mathcal{A}_\mu^a$ and the field strengths $\mathcal{F}_2^a$ of the supergravity multiplet.
The factor $U^{20}(1)$ is associated with $\mathcal{A}_\mu^I$ and the field strength $\mathcal{F}_2^I$ of the Maxwell-matter sector.
Along with the gauge invariant fields strengths

$$\mathcal{F}_3 \quad , \quad \mathcal{F}_2^a \quad , \quad \mathcal{F}_2^I \quad . \tag{3.29}$$

We also have their Poincare duals namely

$$\mathcal{G}_3 \quad , \quad \mathcal{G}_4^a \quad , \quad \mathcal{G}_4^I \quad . \tag{3.30}$$

The $(1+24)$ electric charges and the $(1+24)$ magnetic charges associated with these gauge invariant field strengths are as follows:

(**a**) *dyonic black string BS*:
The 6D dyonic *BS* has an electric charge q_0 and a magnetic charge g_0 with the following quantization condition

$$\mathrm{q}_0\mathrm{g}_0 \quad = \quad 2\pi k_0 \quad , \tag{3.31}$$

where k_0 is an integer ($k_0 \in Z$).

(**b**) *6D black hole BH*:
The six dimensional *BH* has magnetic charges[2] under the gauge symmetry $U^{24}(1)$.

$$\mathrm{g}_\Lambda \quad , \quad \Lambda = 1, ..., 24 \quad . \tag{3.32}$$

(**c**) *6D black membrane BM*:
The six dimensional *BM* is the dual of the black hole and is electrically charged under the $U^{24}(1)$ gauge symmetry:

$$\mathrm{q}_\Lambda \quad , \quad \Lambda = 1, ..., 24 \quad . \tag{3.33}$$

The electric and magnetic charges of the 6D black hole and the 6D black membrane are related by the quantization condition,

$$\mathrm{q}_\Lambda\mathrm{g}_\Lambda \quad = \quad 2\pi k_\Lambda \qquad , \qquad k_\Lambda \in Z \quad . \tag{3.34}$$

[2]For a geometric derivation of the explicit relation between the bare charge g_Λ and the physical charges (m_a, m_I), see [56]

In the brane language of *10D* type IIA superstring on Calabi-Yau manifolds, the electric and/or the magnetic charges are associated with branes wrapping cycles of Calabi-Yau manifold (CY). In the *6D* case we are considering, the CY manifold in question is given by K3 with a homology containing, in addition to the 0- cycle C_0 (K3 points) and the real 4- cycle C_4, real *twenty- two* 2- cycles C_2^I. We also have the following features

6D black attractors	:	electric/magnetic	near horizon geometry	Entropy
dyonic black string	:	$(\mathrm{q}^0, \mathrm{g}_0)$	$AdS_3 \times S^3$	$R_{H_1}^3 G_N^{-\frac{3}{4}}$
black hole	:	$(0, \mathrm{g}_\Lambda)$	$AdS_2 \times S^4$	$R_{H_2}^4 G_N^{-1}$
black membrane	:	$(\mathrm{q}^\Lambda, 0)$	$AdS_4 \times S^2$	$R_{H_3}^2 G_N^{-\frac{1}{2}}$

where the last column stands for the 6D generalized Bekenstein-Hawking entropy formula expressed in terms of the 6D Newton constant G_N and the radius of the horizon geometry. In the case of black string for instance, we have

$$S_{BFS}^{\text{entropy}} \sim G_N^{-\frac{3}{4}} \times \mathcal{A}_{\text{ha}},$$

where $\mathcal{A}_{\text{ha}}$ is the 3d- horizon "hyper-area" in agreement with dimensional arguments and black object thermodynamics laws.
Notice in passing that the *6D* black hole *BH* is made of:
- D0 branes,
- D2 brane wrapping the *twenty- two* 2-cycles of K3, and
- D4 wrapping K3.

The dual black membrane *BM* is made of:
- D2 branes,
- D4 brane wrapping the 2- cycles of K3, and
- D6 brane wrapping K3.

These *6D* black objects have different $AdS_{p+2} \times S^{4-p}$ near horizon geometries; they are schematically represented on the figure 1.

3.3 Entropy of 6D black attractors

Using dimensional arguments and the near horizon geometry, the entropy formula $\mathcal{S}_p$ of the black p-brane attractor in six space time dimensions, that describes the analogue of the Hawking Bekenstein entropy of the *4D* black hole, can be written by as follows

$$\mathcal{S}_p \quad = \quad R_{H_1}^{4-p} G_N^{-\frac{4-p}{4}} \qquad , \qquad p = 0, 1, 2 \quad . \tag{3.35}$$

Here $G_N \sim l_{\text{Planck}}^4$ is the *6D* Newton constant scaling as $(lenght)^4$ and where $R_{H_1}^2$, $R_{H_2}^3$ and $R_{H_3}^4$ stand respectively for the horizon "hyper-areas" of the black hole *BH* $(p = 0)$,

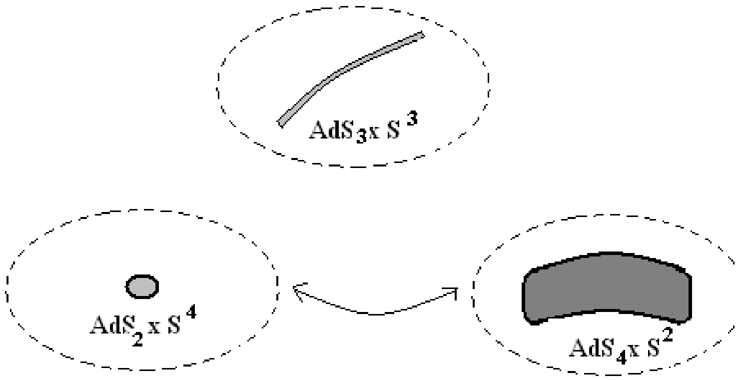

Figure 1: This figure represents the black attractors in 6D N=2 supergravity. Dashed loops refer to the near horizon geometries: (i) On top: we represent a black string with near horizon geometry $AdS_3 \times S^3$. (ii) Bottom-left: a BH with its near horizon $AdS_2 \times S^4$. (iii) Bottom-right: a black membrane with $AdS_4 \times S^2$ geometry .

the dyonic black string BS $(p = 1)$, and the black membrane BM $(p = 2)$.
The entropy $\mathcal{S}_p$ is completely specified by the electric q and magnetic g charges of the black attractor,

$$\mathcal{S}_p \quad = \quad \mathcal{S}_p\left(q, g\right) \quad . \tag{3.36}$$

In the next sections, we will first show that the entropies

$$\begin{array}{lll} \mathcal{S}_{BH} & = & \mathcal{S}_0\left(g_\Lambda\right) \quad , \\ \mathcal{S}_{BS} & = & \mathcal{S}_1\left(q_0, g_0\right) \quad , \\ \mathcal{S}_{BM} & = & \mathcal{S}_2\left(q_\Lambda\right) \quad , \end{array} \tag{3.37}$$

are indeed specified by the appropriate electric q_0 and q_Λ as well as the corresponding magnetic g_0 and g_Λ ones.
Then we use this result to check explicitly that, at supergravity level, the entropy $\mathcal{S}_1\left(q_0, g_0\right)$ of the 6D black string is invariant under electric/magnetic duality.
This property is also used to conjecture that the invariance of $\mathcal{S}_{BS}$ under the electric/magnetic duality is a *general feature* of dyonic objects including black brane bound states.
As such, invariance under electric/magnetic duality should holds also for the entropy

$$\mathcal{S}_{EBB-MBB} \quad = \quad \mathcal{S}_{DP}\left(q_\Lambda, g_\Lambda\right) \quad , \tag{3.38}$$

of the dual pair bounds $DP \equiv$ (*EBB-MBB*) given by eqs(3.9,3.13,3.14).
By using this natural conjecture, it follows that under the dual change

$$\begin{array}{lll} q_A & \rightarrow & q'_A = g_A \quad , \\ g_A & \rightarrow & g'_A = q_A \quad , \end{array} \tag{3.39}$$

we should also have

$$\mathcal{S}_{EBB}\left(q_A\right) \quad \leftrightarrow \quad \mathcal{S}_{MBB}\left(g^A\right) \quad , \tag{3.40}$$

where $\mathcal{S}_{EBB}$ and $\mathcal{S}_{MBB}$ stand respectively for the entropies of an electrically charged black brane EBB and its magnetic dual MBB.
In this view, invariance of the entropy $\mathcal{S}_1$ of the dyonic black string BS under the change (3.39) as well as of the dual pair bounds DP,

$$\begin{aligned} \mathcal{S}_1 \left(q'_0, g'_0 \right) &= \mathcal{S}_1 \left(q_0, g_0 \right) \quad , \\ \mathcal{S}_{DP} \left(q'_\Lambda, g'_\Lambda \right) &= \mathcal{S}_{DP} \left(q_\Lambda, g_\Lambda \right) \quad , \end{aligned} \tag{3.41}$$

follows straightforwardly. The result for the case $\mathcal{S}_{DP} \left(q_\Lambda, g_\Lambda \right)$ will be explicitly derived in section 6. The case of the *6D* black string BS will be studied below.

4. Black string

The six dimensional black string is a dyonic black attractor solution of the $\mathcal{N} = 2$ non chiral supergravity with near horizon geometry $AdS_3 \times S^3$. The magnetic charge $m = \mathrm{g}^0$ and the electric charge $e = q_0$ carried by the BS are those of the gauge invariant 3- form field strengths $\mathcal{F}_3$ and $\mathcal{G}_3 = {}^\star\mathcal{F}_3$. The $\star$ conjugation stands for the usual six dimensional Hodge duality interchanging n- forms with $(6 - n)$ ones.
The field strengths $\mathcal{F}_3$ and $\mathcal{G}_3$ are associated with the NS-NS 2- form $\mathcal{B}_{\mu\nu}$ fields in six dimensions. Using eqs(3.22), we have

$$\mathrm{g}_0 = \int_{S^3} \mathcal{F}_3 \qquad , \qquad \mathrm{q}^0 = \int_{S^3} \mathcal{G}_3 \quad . \tag{4.1}$$

The electric q^0 and magnetic g_0 charges obey the quantization condition (3.31).

4.1 Entropy of the black string

The effective potential $\mathcal{V}_{BS} = \mathcal{V}_{BS} \left(\sigma, \mathrm{g}_0, \mathrm{q}_0 \right)$ of the BS is given by the the *6D* extension of the *4D Weinhold* relation [46, 53, 54]. It reads in terms of the dilaton field $\sigma = \sigma \left(x \right)$ of the 6D $\mathcal{N} = 2$ supergravity multiplet and the electric/magnetic charges like [55],

$$\mathcal{V}_{BS} \quad = \quad \tfrac{\mathrm{g}_0^2}{2} \exp \left(-4\sigma \right) \quad + \quad \tfrac{\mathrm{q}_0^2}{2} \exp \left(4\sigma \right) \quad . \tag{4.2}$$

In addition to the exponential behavior, the potential $\mathcal{V}_{BS}$ has the remarkable invariance under the following change,

$$\sigma \rightarrow -\sigma \qquad \text{and} \qquad \mathrm{g}_0 \leftrightarrow \mathrm{q}_0 \quad . \tag{4.3}$$

At the horizon $r = R^{BS}_{\text{horizon}}$ of the dyonic BS, the above potential $\mathcal{V}_{BS}$ is at its minimum. The value of the dilaton at horizon

$$\sigma_1 = \sigma \left(r = R^{BS}_{\text{horizon}} \right) , \tag{4.4}$$

is obtained by solving the following constraint equation

$$\frac{d\mathcal{V}_{BS} \left(\sigma \right)}{d\sigma} = 0, \tag{4.5}$$

which in turns leads to,

$$2\mathrm{q}_0^2 \exp(4\sigma) \quad - \quad 2\mathrm{g}_0^2 \exp(-4\sigma) \quad = 0 \quad . \tag{4.6}$$

The critical value σ_1 of the dilaton at the horizon R^{BS}_{horizon} is given by

$$\exp(4\sigma_1) \quad = \quad \pm\tfrac{\mathrm{g}_0}{\mathrm{q}_0} > 0 \quad , \tag{4.7}$$

or equivalently

$$\sigma_1 \quad = \quad \tfrac{1}{4}\ln\left(\left|\tfrac{\mathrm{g}_0}{\mathrm{q}_0}\right|\right) \quad . \tag{4.8}$$

From this solution, we learn two interesting information:
The first information, noted previously in [56], concerns the electric/magnetic duality. The latter requires interchanging the electric q_0 and magnetic g_0 charges; but also performing the change

$$\sigma \rightarrow -\sigma \tag{4.9}$$

in the moduli space. This property is manifestly exhibited by eqs(4.3-4.6).
The second information we learn concerns the critical value σ_1 of the dilaton at the horizon of the black string eq(4.8). Finite critical values σ_1 of the dilaton requires that both the electric q_0 and the magnetic g_0 charges have to be non zero, i.e

$$\mathrm{q}_0\mathrm{g}_0 \neq 0 \ , \tag{4.10}$$

or equivalently by using eq(3.31)

$$\mathrm{k}_0 \neq 0 \ . \tag{4.11}$$

We will see later that this is a general property valid also for of the *6D* dyonic pair *BH-BM*. Moreover, the value $\mathcal{V}^{\min}_{BS}$ of the BS potential $\mathcal{V}_{BS}(\sigma)$ at the minimum of the black string potential is

$$\mathcal{V}^{\min}_{BFS}(\sigma_1) = |\mathrm{q}_0\mathrm{g}_0| > 0 \ , \tag{4.12}$$

and so the BS entropy reads as

$$\mathcal{S}_1 = \frac{|\mathrm{q}_0\mathrm{g}_0|}{4} \ . \tag{4.13}$$

Up on using eq(3.31), $\mathcal{S}_1$ can be also expressed as

$$\mathcal{S}_1 = \pi\frac{|k_0|}{2} \ . \tag{4.14}$$

where k_0 is an integer. Notice that because of the constraint eq(4.10), the entropy $\mathcal{S}_1$ given by the above eqs is necessarily positive definite.
Below, we want to discuss what happens to the entropy $\mathcal{S}_1$ if we try to go beyond the constraint eq(4.10).

4.2 Regular and singular representations

For later use, we make two comments concerning the effective potential of the dyonic black string. The first comment concerns the generic case where $k_0 \neq 0$ and the second deals with the singular case $k_0 = 0$.

(**1**) *case* $k_0 \neq 0$:

This case corresponds to the dyonic black string of the 6D $\mathcal{N} = 2$ non chiral supergravity we have been studding. In fact it is interesting to distinguish two situations:

(a) *Regular representation:* $\mathrm{g}_0 \neq 0$, $\mathrm{q}_0 \neq 0$

Here, the extremum of the black string potential at

$$\sigma_1 = \frac{1}{4} \ln \left(\left| \frac{\mathrm{g}_0}{\mathrm{q}_0} \right| \right) \quad , \tag{4.15}$$

is well defined and is precisely a minimum. Since g_0 and q_0 are related as in eq(3.31), we can be expressed σ_1 either as

$$\sigma_1 = \frac{1}{4} \ln \left| \frac{2\pi k_0}{\mathrm{q}_0^2} \right| \quad , \tag{4.16}$$

by using the electric charge q_0 and the integer k_0, or equivalently by using the magnetic charge g_0 like

$$\sigma_1 = \frac{1}{4} \ln \left| \frac{\mathrm{g}_0^2}{2\pi k_0} \right| \quad . \tag{4.17}$$

The value of the potential at the minimum depends remarkably on the integer k_0 as shown below,

$$\mathcal{V}_{BFS}^{\min} (\sigma_1) = 2\pi \left| k_0 \right| > 0 \ . \tag{4.18}$$

This is an interesting property that let understand that a non zero entropy value seems to need dyonic charged black branes since if taking for example $\mathrm{g}_0 \neq 0$ and finite but $\mathrm{q}_0 = 0$, eq(4.18) vanishes identically.

(b) *Singular representation*

Notice that, strictly speaking, the condition $k_0 \neq 0$ corresponds to $\mathrm{g}_0\mathrm{q}_0 \neq 0$. But this condition could be solved in general in two ways:

First by using regular finite charges $\mathrm{g}_0 \neq 0$ and $\mathrm{q}_0 \neq 0$ as just discussed above.

Second by considering the singular situation where we have an infinite number of electric charges and no magnetic charge; i.e

$$\mathrm{q}_0 \to \infty \qquad \text{and} \qquad \mathrm{g}_0 \to 0, \tag{4.19}$$

together with the constraint $\mathrm{g}_0 = \frac{k_0}{\mathrm{q}_0}$.

We can also have the symmetric case where we do have an infinite number of magnetic charges and no electric charge:

$$\mathrm{q}_0 \to 0 \qquad \text{and} \qquad \mathrm{g}_0 \to \infty. \tag{4.20}$$

These particular and singular configurations are in some sense formal; but very suggestive. They will be used later on to approach the dyonic pair *BH-BM*.

(2) *Case* $k_0 = 0$:
Using eq(3.31), this case is solved as

$$\mathrm{g}_0 \neq 0 \qquad , \qquad \mathrm{q}_0 = 0. \tag{4.21}$$

or like

$$\mathrm{g}_0 = 0 \qquad , \qquad \mathrm{q}_0 \neq 0. \tag{4.22}$$

They can be also associated with the (self dual and anti-self dual part) black string of the 6D $\mathcal{N} = (2, 0)$ chiral supergravity. There, the *NS-NS* B- field field $\mathcal{B}_{[\mu\nu]}$ splits into a self dual part

$$\mathcal{B}^{+}_{[\mu\nu]} \ , \tag{4.23}$$

and anti-self dual part

$$\mathcal{B}^{-}_{[\mu\nu]} \ . \tag{4.24}$$

The strength $\mathcal{H}^{+}_{[\lambda\mu\nu]}$ associated with the self dual part $B^{+}_{[\mu\nu]}$,

$$\mathcal{H}^{+}_{3} = dB^{+}_{2}, \tag{4.25}$$

is in the gravity supermultiplet while the field strength $\mathcal{H}^{-}_{\lambda\mu\nu}$ of the anti-self dual part $\mathcal{B}^{-}_{[\mu\nu]}$,

$$\mathcal{H}^{-}_{3} = dB^{-}_{2}, \tag{4.26}$$

together with the field σ, are in the tensor multiplet.
The minimum $\mathcal{V}^{\min}_{BS}(\sigma_1)$ of the black string potential is at infinity; that is either at

$$\sigma_1 \rightarrow +\infty \ , \tag{4.27}$$

or at

$$\sigma_1 \rightarrow -\infty \ .$$

In both cases, $\mathcal{V}^{\min}_{BS}(\sigma_1)$ takes a zero value in agreement with eq(4.18). For illustration; we give in *figure 2* the general behavior of the black string potential in terms of the dilaton σ.

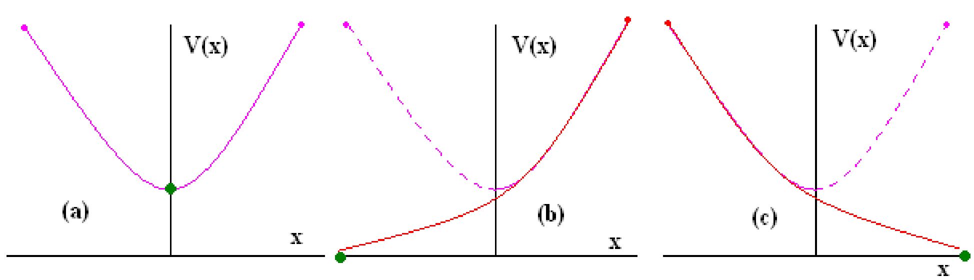

Figure 2: Variation of the effective potential with to the dilaton field. (a) Case of black string in *6D* $\mathcal{N} = 2$ non chiral supergravity. (b/c) black string in *6D* $\mathcal{N} = (2, 0)$ chiral supergravity.

4.3 Electric/magnetic duality

The key property of the above dyonic black string entropy relation (4.13) is that $\mathcal{S}_{BS}$ is invariant under the following electric-magnetic change

$$\begin{array}{ccc} \mathrm{q}_0 & \rightarrow & \mathrm{q}_0' = \mathrm{g}_0 \quad , \\ \mathrm{g}_0 & \rightarrow & \mathrm{g}_0' = \mathrm{q}_0 \quad , \end{array} \tag{4.28}$$

with

$$\mathcal{S}_{BS}\left(\mathrm{q}_0, \mathrm{g}_0\right) \quad \rightarrow \quad \mathcal{S}_{BS}\left(\mathrm{q}_0', \mathrm{g}_0'\right) = \mathcal{S}_{BS}\left(\mathrm{q}_0, \mathrm{g}_0\right) \quad . \tag{4.29}$$

This property, which can be explicitly checked on previous eqs, was expected since we are dealing with a dyonic object.
Moreover, using the discussion of sub-section 4.2 (*singular representation*), the relation (4.29) can be extended to the 6D dyonic pair *BH-BM*.

Black hole/black membrane
Denoting by $\mathcal{S}_{BH}$ the entropy of magnetically charged black hole *BH*; i.e

$$\mathcal{S}_{BH} = \mathcal{S}_{BH}\left(\mathrm{g}_\Lambda\right) \quad , \tag{4.30}$$

and by $\mathcal{S}_{BM}$ the entropy of the electrically charged black membrane *BM*; i.e

$$\mathcal{S}_{BM} = \mathcal{S}_{BM}\left(\mathrm{q}_\Lambda\right) \quad . \tag{4.31}$$

Then performing the electric/magnetic duality (4.28), the entropies $\mathcal{S}_{BH}\left(\mathrm{g}_\Lambda\right)$ and $\mathcal{S}_{BM}\left(\mathrm{q}_\Lambda\right)$ are interchanged as follows

$$\begin{array}{ccc} \mathcal{S}_{BH}\left(\mathrm{g}_\Lambda\right) & \rightarrow & \mathcal{S}_{BH}\left(\mathrm{g}_\Lambda'\right) = \mathcal{S}_{BM}\left(\mathrm{q}_\Lambda\right) \quad , \\ \mathcal{S}_{BM}\left(\mathrm{q}_\Lambda\right) & \rightarrow & \mathcal{S}_{BM}\left(\mathrm{q}_\Lambda'\right) = \mathcal{S}_{BH}\left(\mathrm{g}_\Lambda\right) \quad . \end{array} \tag{4.32}$$

From these relations we learn that, like for the dyonic *BS* singlet, the entropy

$$\mathcal{S}_{DP} = \mathcal{S}_{BH-BM}\left(\mathrm{g}_\Lambda, \mathrm{q}_\Lambda\right), \tag{4.33}$$

of the dyonic pair $DP \equiv$ *BH-BM* obeys the same identity as eq(4.29). It is invariant under electric/magnetic duality transformation.

$$\mathcal{S}_{DP}\left(\mathrm{g}_\Lambda, \mathrm{q}_\Lambda\right) \quad = \quad \mathcal{S}_{DP}\left(\mathrm{g}_\Lambda', \mathrm{q}_\Lambda'\right) \quad . \tag{4.34}$$

In what follows, we want to prove this statement by computing explicitly the expressions of $\mathcal{S}_{DP}\left(\mathrm{g}_\Lambda, \mathrm{q}_\Lambda\right)$ and $\mathcal{S}_{DP}\left(\mathrm{g}_\Lambda', \mathrm{q}_\Lambda'\right)$. But before that we need to study the effective potentials and the attractor eqs of the following:
(**a**) the *BPS* and non *BPS* black holes in *6D* $\mathcal{N} = 2$ non chiral supergravity
(**b**) the *BPS* and non *BPS* black membranes in *6D* $\mathcal{N} = 2$ non chiral supergravity
(**c**) the *BPS* and non *BPS* dyonic pairs $DP \equiv$ (*BH-BM*).
The effective potential and the attractor mechanism of the *6D* black string *BS* has been explicitly studied in [55]. We will then just give the results.

We also take this opportunity to develop a *new method* to deal with the computation of the critical values of the effective potentials of the BH and the BM in six dimensions. This new method relies on enlarging the moduli space of the *6D* $\mathcal{N}=2$ supergravity by including Lagrange multipliers,

$$\lambda^{\Lambda\Sigma}=\lambda^{\Sigma\Lambda}, \tag{4.35}$$

capturing the constraint eqs on the field matrix

$$L_{\Lambda\Sigma}=L_{\Lambda\Sigma}\left(x\right), \tag{4.36}$$

used to parameterize the $SO\left(4,20\right)$ orthogonal group manifold involved in the moduli space $\frac{SO(4,20)}{SO(4)\times SO(20)}$.

5. Attractor eqs and Lagrange multiplier method

We first describe the effective potential $\mathcal{V}_{BH}$ of the *6D* black hole. Then, we use the electric/magnetic duality and results from [56] to determine the effective potential $\mathcal{V}_{BM}$ of the black membrane. After that, we study the attractor eqs and their solutions by combining the approach of the criticality of the potential and the Lagrange multiplier method.

5.1 Effective potential $\mathcal{V}_{BH}$

In the *6D* $\mathcal{N}=2$ non chiral supergravity embedded in *10D* type IIA superstring on K3, the black hole BH is magnetically charged under the $U^{4}\left(1\right)\times U^{20}\left(1\right)$ gauge group symmetry.
The bare magnetic charges g^{Λ} are given by,

$$\mathrm{g}^{\Lambda} \quad = \quad \int_{S^2}\mathcal{F}_2^{\Lambda} \quad , \quad \Lambda=1,...,24 \quad . \tag{5.1}$$

The magnetic charges g^{Λ} form a charge vector the group $SO\left(4,20\right)$ with signature *4*$(+)$ and *20*$(-)$ captured by the diagonal flat metric $\eta_{\Lambda\Sigma}$ of the tangent space $\mathbb{R}^{(4,20)}$.

5.1.1 Potential of the BH

The effective scalar potential $\mathcal{V}_{BH}$ of the black hole is given by the Weinhold relation, expressed in the flat coordinate frame,

$$\mathcal{V}_{BH}=\left(\sum_{a=1}^{4}\delta_{ab}Z^{a}Z^{b}+\sum_{I=1}^{20}\delta_{IJ}Z^{I}Z^{J}\right). \tag{5.2}$$

In this relation, the central charges Z_a and Z_I are respectively the *dressed* charges describing respectively the physical charges of the *four* Maxwell fields in the gravity supermultiplet and the *twenty* Maxwell fields of the matter sector.
The dressing of the charges is given by the following linear combination,

$$\begin{array}{lll} Z_a & = & \sum_{\Lambda=1}^{24}U_{a\Lambda}\mathrm{g}^{\Lambda} \quad , \\ Z_I & = & \sum_{\Lambda=1}^{24}U_{I\Lambda}\mathrm{g}^{\Lambda} \quad , \end{array} \tag{5.3}$$

where $U_{\Sigma\Lambda}$ parameterize the moduli space $\boldsymbol{M}_{6D}^{N=2}$ of *10D* type IIA superstring on K3,

$$\begin{array}{lcl} \boldsymbol{M}_{6D}^{N=2} & = & SO\left(1,1\right)\times G_{6} \quad , \\ G_{6} & = & \frac{SO(4,20)}{SO(4)\times SO(20)} \quad . \end{array} \tag{5.4}$$

Notice that the real matrix $U_{\Lambda\Sigma}$ obeys a set of constraint relations that can be used to put $U_{\Lambda\Sigma}$ in a more convenient form. We have the following properties:
(**i**) the factorization property which allows to factorize $U_{\Lambda\Sigma}$ as follows:

$$\begin{array}{lcl} U_{\Lambda\Sigma} & = & e^{-\sigma}L_{\Lambda\Sigma} \quad , \\ U_{\Lambda\Sigma}^{-1} & = & e^{\sigma}L_{\Lambda\Sigma}^{-1} \quad , \\ L^{-1} & = & \eta L^{t}\eta \quad . \end{array} \tag{5.5}$$

Here $e^{-\sigma}$ parameterizes the factor $SO\left(1,1\right)$ of $\boldsymbol{M}_{6D}^{N=2}$ and $L_{\Lambda\Sigma}$ defines G_6. Multiplying this equation by the magnetic charge vector g^{Σ}, we obtain the dressed magnetic charge vector $Z_{\Lambda}=(Z_a,Z_I)$,

$$\begin{array}{lcccl} Z_{a} & = & \sum\limits_{\Sigma=1}^{24}U_{a\Sigma}\mathrm{g}^{\Sigma} & = & e^{-\sigma}\sum\limits_{\Sigma=1}^{24}L_{a\Sigma}\mathrm{g}^{\Sigma} \quad , \\ Z_{I} & = & \sum\limits_{\Sigma=1}^{24}U_{I\Sigma}\mathrm{g}^{\Sigma} & = & e^{-\sigma}\sum\limits_{\Sigma=1}^{24}L_{I\Sigma}\mathrm{g}^{\Sigma} \quad . \end{array} \tag{5.6}$$

(**ii**) the *orthogonality* property of the elements of the $SO\left(4,20\right)$ group which requires that the real 24×24 matrices $L_{\Lambda\Sigma}$ should obey the orthogonality condition:

$$\sum_{c,d=1}^{4}\delta^{cd}L_{c\Lambda}L_{d\Sigma} \quad - \quad \sum_{K,L=1}^{20}\delta^{KL}L_{K\Lambda}L_{L\Sigma} \quad = \quad \eta_{\Lambda\Sigma} \quad , \tag{5.7}$$

with Λ and $\Sigma=1,\cdots,24$. This relation can be also rewritten as

$$\left(L^{t}\eta L\right)_{\Lambda\Sigma} \quad = \quad \sum_{\Upsilon,\Gamma=1}^{24}L_{\Lambda}^{\Upsilon}\eta_{\Upsilon\Gamma}L_{\Sigma}^{\Gamma} \quad = \quad \eta_{\Lambda\Sigma} \quad . \tag{5.8}$$

Multiplying both sides of this equation by $\mathrm{g}^{\Lambda}\mathrm{g}^{\Sigma}$, we obtain the following constraint eq

$$\sum_{a,b=1}^{4}\delta_{ab}Z^{a}Z^{b}-\sum_{I,J=1}^{20}\delta_{IJ}Z^{I}Z^{J} \quad = \quad e^{-2\sigma}\mathrm{g}^{2} \quad , \tag{5.9}$$

with

$$\sum_{\Lambda,\Sigma=1}^{24}\mathrm{g}^{\Lambda}\eta_{\Lambda\Sigma}\mathrm{g}^{\Sigma} \quad = \quad \mathrm{g}^{2} \quad . \tag{5.10}$$

This relation expresses the orthogonality condition in terms of the magnetic charges. We will refer to it as the "magnetic orthogonality" relation.
(**iii**) the isotropy invariance under $SO\left(4\right)\times SO\left(20\right)$ which acts on the matrix $L_{\Lambda\Sigma}\in SO\left(4,20\right)$ as a gauge group symmetry,

$$L=hLh^{-1}, \tag{5.11}$$

where $h\in SO\left(4\right)\times SO\left(20\right)$, the maximal compact subgroup of $SO\left(4,20\right)$.

5.1.2 Implementing the Lagrange multiplier

Notice that the matrix variable $L_{\Lambda\Sigma}$ has 24×24 real parameters which is much larger that the *80* moduli required by (5.4). The properties (**ii**) and (**iii**) are then constraint eqs on $L_{\Lambda\Sigma}$ which is convenient to cast as follows:

$$L_{\Lambda\Sigma} = \begin{pmatrix} L_{ab} & L_{aJ} \\ L_{Ib} & L_{IJ} \end{pmatrix}. \tag{5.12}$$

To deal with the undesired degrees of freedom in $L_{\Lambda\Sigma}$, we proceed as follows:
(**1**) we fix the $SO\,(4) \times SO\,(20)$ gauge symmetry by working in the gauge where the sub-matrices L_{ab} and L_{IJ} are taken symmetric:

$$L_{ab} = L_{ba} \qquad , \qquad L_{IJ} = L_{JI}. \tag{5.13}$$

(**2**) the orthogonality property eq(5.8)

$$\left(L^t \eta L\right)_{\Lambda\Sigma} = \eta_{\Lambda\Sigma} \tag{5.14}$$

will be imposed by using the Lagrange multiplier method. This method should be understood as an alternative way to the usual Maurer-Cartan equation generally used to deal with this matter [46].
Eq(5.8) suggests that the Lagrange multipliers should be a symmetric matrix field $\lambda^{\Lambda\Sigma}$ like in eq(4.35); but the equivalent reduced form eq(5.9) of the constraints suggests that it is more convenient to take the Lagrange parameters $\lambda^{\Lambda\Sigma}$ as follows,

$$\lambda^{\Lambda\Sigma} = \lambda g^{\Lambda} g^{\Sigma}, \tag{5.15}$$

where now we have only one Lagrange multiplier λ.
Therefore the previous expression of the effective scalar potential $\mathcal{V}_{BH}$ of the black hole potential can be put into the following form

$$\widetilde{\mathcal{V}}_{BH}\,(\sigma, Z, \lambda) = \left(Z_a Z^a + Z_I Z^I\right) + \lambda \left(Z_a Z^a - Z_I Z^I - e^{-2\sigma} \mathrm{g}^2\right)\ , \tag{5.16}$$

where we have set

$$\begin{aligned} Z_a Z^a &= \sum_{a,b=1}^{4} \delta_{ab} Z^a Z^b \quad , \\ Z_I Z^I &= \sum_{I,J=1}^{20} \delta_{IJ} Z^I Z^J \quad . \end{aligned} \tag{5.17}$$

In this relation, we have an extra dependence on the Lagrange multiplier λ. Moreover, setting

$$\begin{aligned} Z_a &= e^{-\sigma} R_a \quad , \\ Z_I &= e^{-\sigma} R_I \quad , \end{aligned} \tag{5.18}$$

with

$$\begin{aligned} R_a &= L_{a\Lambda} \mathrm{g}^{\Lambda} \quad , \\ R_I &= L_{I\Lambda} \mathrm{g}^{\Lambda} \quad , \end{aligned} \tag{5.19}$$

we can factorize out the dilaton field dependence in the effective potential. We have:

$$\widetilde{\mathcal{V}}_{BH}(\sigma, R, \lambda) = e^{-2\sigma}\mathcal{V}_0(R, \lambda, \mathrm{g}) \tag{5.20}$$

with $\mathcal{V}_0(R, \lambda, \mathrm{g})$,

$$\mathcal{V}_0(R, \lambda, \mathrm{g}) = \left(R_a R^a + R_I R^I\right) + \lambda\left(R_a R^a - R_I R^I - \mathrm{g}^2\right), \tag{5.21}$$

being the potential of the black hole at $\sigma = 0$. The factor $\mathcal{V}_0$ has no dependence in the dilaton field.

5.1.3 Attractor eqs and their solutions

Because of the structure of the effective black hole potential

$$\widetilde{\mathcal{V}}_{BH}(Z, \lambda) \quad = \quad \widetilde{\mathcal{V}}_{BH}(\sigma, R, \lambda) \quad , \tag{5.22}$$

with

$$\begin{array}{lcll} Z & = & Z(\sigma, R) & , \\ R & = & R(L_{\Lambda\Sigma}) & , \end{array} \tag{5.23}$$

the attractor eqs for the six dimensional magnetic black hole can be written in different, but equivalent manners depending of the variables we use.
For example, the attractor eqs can stated by using as variables the dilaton σ, the dressed charges R^a and R^I and obviously the Lagrange multiplier λ. Then we have,

$$\begin{array}{lcll} \frac{\delta\widetilde{\mathcal{V}}_{BH}}{\delta\sigma} & = & 0 & , \\ \frac{\delta\widetilde{\mathcal{V}}_{BH}}{\delta R} & = & 0 & , \\ \frac{\delta\widetilde{\mathcal{V}}_{BH}}{\delta\lambda} & = & 0 & . \end{array} \tag{5.24}$$

They can be also expressed by using as variables the dressed central charges $Z^a = e^{-\sigma}R^a$ and $Z^I = e^{-\sigma}R^I$ and the Lagrange multiplier as follows,

$$\begin{array}{lcll} (1+\lambda)\,Z^a & = & 0 & , \\ (1-\lambda)\,Z^I & = & 0 & , \\ R_a R^a - R_I R^I & = & \mathrm{g}^2 & , \end{array} \tag{5.25}$$

where now the dilaton has been absorbed in the Z's.
There are three kinds of solutions of the eqs(5.25). These solutions are given by:

$$\text{solution (1)}: \quad Z_a = 0 \quad , \quad Z_I = 0 \quad , \quad \lambda \neq \pm 1 \, , \tag{5.26}$$

$$\text{solution (2)}: \quad Z_a = 0 \quad , \quad Z_I \neq 0 \quad , \quad \lambda = +1 \, , \tag{5.27}$$

$$\text{solution (3)}: \quad Z_a \neq 0 \quad , \quad Z_I = 0 \quad , \quad \lambda = -1 \, . \tag{5.28}$$

The first one is a singular degenerate solution; while the two others describe respectively a non BPS and a BPS black hole.
Moreover, since the dressed magnetic charge Z is given by the product $e^{-\sigma}R$, the vanishing of the product

$$e^{-\sigma}R = 0 \, , \tag{5.29}$$

can, in addition to the singular case $e^{-\sigma} = 0 = R$, be solved as well by taking

$$e^{-\sigma} = 0 \quad , \quad R \neq 0 \tag{5.30}$$

or

$$e^{-\sigma} \neq 0 \quad , \quad R = 0. \tag{5.31}$$

Then, we have to distinguish the two following solutions:

(a) *case 1*: $\sigma_0 \to \infty$

In this case, the value of the minimum of the potential at the critical point is given by,

$$\widetilde{\mathcal{V}}_{BH}^{\min} = e^{-2\sigma_0} \mathrm{g}^2 = 0. \tag{5.32}$$

So the entropy $\mathcal{S}_{BH}$ of the black hole is zero,

$$\mathcal{S}_{BH} = 0. \tag{5.33}$$

This configuration corresponds to the solution (5.26).

(b) *case 2*: $\sigma = \sigma_0 < \infty$.

Here the value of the dilaton at horizon $\sigma(r_{\text{horizon}}) = \sigma_0$ is given by a *finite* number ($\sigma_0 < \infty$). The two solutions (5.27-5.28) read as follows

$$\begin{array}{llllllll} \text{case } (2a): & R_a = 0 & , & R_I \neq 0 & , & \lambda = 1 & , \\ \text{case } (2b): & R_a \neq 0 & , & R_I = 0 & , & \lambda = -1 & . \end{array} \tag{5.34}$$

The corresponding values of the 6D black potential $\widetilde{\mathcal{V}}_{BH}^{\min}$ at the minimum are

$$\begin{array}{lllll} \text{case } (2a): & \widetilde{\mathcal{V}}_{BH}^{\min} = -e^{-2\sigma_0} \mathrm{g}^2 > 0 & , & \mathrm{g}^2 < 0 & , \\ \text{case } (2b): & \widetilde{\mathcal{V}}_{BH}^{\min} = +e^{-2\sigma_0} \mathrm{g}^2 > 0 & , & \mathrm{g}^2 > 0 & . \end{array} \tag{5.35}$$

In these relations, σ_0 is a *free parameter*. To fix it, we need an extra constraint. We will see later on that σ_0 can be indeed fixed in the case of the dyonic pair *DP* $\equiv$ *BH-BM*. Before that let us complete this analysis by considering also the effective potential $\mathcal{V}_{BM}$ of the black membrane and its entropy $\mathcal{S}_{BM}$.

5.2 Effective potential $\mathcal{V}_{BM}$

The electrically charged black membrane *BM* is the dual of the magnetic black hole *BH* considered above. Its effective scalar potential $\mathcal{V}_{BM}$ depends on the electric charges q_Λ and the field variables of the moduli space (5.4). The 24 bare electric charges q_Λ are given by,

$$\begin{array}{lll} \mathrm{q}^\Lambda & = & \int_{S^4} \mathcal{F}_4^\Lambda \quad , \\ \mathcal{F}_4^\Lambda & = & {}^\star\left(\mathcal{F}_2^\Lambda\right) \quad . \end{array} \tag{5.36}$$

The q_Λ's with $\Lambda = 1, ..., 24$, form a 24- vector charge of the group $SO(4,20)$.

The explicit expression of potential $\mathcal{V}_{BM}$ of the black membrane can be read like

$$\mathcal{V}_{BM} = \left(W_a W^a + W_I W^I\right) \ , \tag{5.37}$$

where now W_a and W_I are respectively the dressed electric charges of the bare ones q_Λ. These dressed charges can be expressed as linear combination as follows,

$$\begin{aligned} W^a &= \sum_{\Lambda=1}^{24} \mathrm{q}_\Lambda P^{\Lambda a} \quad , \\ W^I &= \sum_{\Lambda=1}^{24} \mathrm{q}_\Lambda P^{\Lambda I} \quad , \end{aligned} \tag{5.38}$$

where, like for $U_{\Lambda\Sigma}$ of eq(5.3), the field matrix $P^{\Lambda\Sigma}$ parameterizes the moduli space (5.4). The matrices $P^{\Lambda\Sigma}$ and $U_{\Lambda\Sigma}$ are then related to each other. To obtain this relation, we use the electric/magnetic duality exchanging the black hole and the black membrane charges. Formally, the electric/magnetic duality can be stated at the level of the effective scalar potentials $\mathcal{V}_{BH}$ and $\mathcal{V}_{BM}$ like

$$\begin{aligned} \mathrm{g}^\Lambda &\leftrightarrow \mathrm{q}_\Lambda \quad , \\ \mathcal{V}_{BH} &\leftrightarrow \mathcal{V}_{BM} \quad , \end{aligned} \tag{5.39}$$

and then

$$\begin{aligned} R_a &\leftrightarrow W^a \quad , \\ R_I &\leftrightarrow W^I \quad . \end{aligned} \tag{5.40}$$

Extending the electric/magnetic duality relation eqs(3.33), which we rewrite as follows,

$$\begin{aligned} \textstyle\sum_{\Lambda=1}^{24} \mathrm{q}_\Lambda \mathrm{g}^\Lambda &= 2\pi k \quad , \\ \textstyle\sum_{\Lambda=1}^{24} k_\Lambda &= k \in \mathbb{Z} \quad , \end{aligned} \tag{5.41}$$

to the dressed charges,

$$\sum_{\Lambda=1}^{24} W^\Lambda Z_\Lambda = k \ , \tag{5.42}$$

we can determine the relation between $U_{\Lambda\Sigma}$ and $P_{\Lambda\Sigma}$ matrices. Indeed putting

$$W^\Lambda = \sum_\Sigma P^{\Lambda\Sigma} \mathrm{q}_\Sigma, \tag{5.43}$$

and

$$Z_\Lambda = \sum_\Upsilon \mathrm{g}^\Upsilon L_{\Upsilon\Lambda}, \tag{5.44}$$

back into above relation, we find that the matrix $P^{\Upsilon\Lambda}$ is just the inverse of the matrix $U_{\Sigma\Upsilon}$.

Therefore, electric/magnetic duality mapping the black hole BH to the black membrane BM is given by

black hole BH	electric/magnetic $\longleftrightarrow$	black membrane BM
g^{Λ}	$\leftrightarrow$	q_{Λ}
$U_{\Sigma\Upsilon}$	$\leftrightarrow$	$P^{\Upsilon\Lambda} = (U_{\Sigma\Upsilon})^{-1}$
Z_{Λ}	$\leftrightarrow$	W^{Λ}
λ	$\leftrightarrow$	ξ

(5.45)

Notice that, like for the matrix $U_{\Lambda\Sigma}$, we also have the following properties:
(**i**) the factorization of the moduli space (5.4) as $SO(1,1) \times \frac{SO(4,20)}{SO(4)\times SO(20)}$ allows to factorize $P^{\Lambda\Sigma}$ like,

$$P^{\Lambda\Sigma} = e^{+\sigma} \left(L^{-1}\right)^{\Lambda\Sigma}. \tag{5.46}$$

Multiplying both sides of this equation by q_{Σ}, we obtain the dressed central charges $W^{\Sigma} = \left(W^{a}, W^{I}\right)$

$$\begin{array}{lll} W^{a} & = & q_{\Lambda} P^{\Lambda a} \quad , \\ W^{I} & = & q_{\Lambda} P^{\Lambda I} \quad . \end{array} \tag{5.47}$$

As in the case of 6D black hole eq(5.18), these charges factorize as well like

$$\begin{array}{lll} W^{a} & = & e^{+\sigma} T^{a} \quad , \\ W^{I} & = & e^{+\sigma} T^{I} \quad , \end{array} \tag{5.48}$$

with

$$\begin{array}{lll} T^{a} & = & q_{\Lambda} \left(L^{-1}\right)^{\Lambda a} \quad , \\ T^{I} & = & q_{\Lambda} \left(L^{-1}\right)^{\Lambda I} \quad . \end{array} \tag{5.49}$$

The dressed charges T^{a} and T^{I} are the dual of the R_{a} and R_{I}.
(**ii**) the orthogonality property of the non compact $SO(4,20)$ group, which we can be written as

$$\left(\sum_{a=1}^{4} \left(L^{-1}\right)^{a\Lambda} \left(L^{-1}\right)^{a\Sigma} - \sum_{K=1}^{20} \left(L^{-1}\right)^{K\Lambda} \left(L^{-1}\right)^{K\Sigma}\right) = \eta^{\Lambda\Sigma}, \tag{5.50}$$

allows to get more information on the dressed electric charges. Multiplying both sides of this algebraic constraint equation by $q_{\Lambda}q_{\Sigma}$, we obtain

$$W_{a}W^{a} - W_{I}W^{I} = e^{+2\sigma} q^{2} \ , \tag{5.51}$$

with

$$q^{2} = q_{\Lambda}\eta^{\Lambda\Sigma}q_{\Sigma} = \left(\sum_{a=1}^{4} q_{a}^{2} - \sum_{I=1}^{20} q_{I}^{2}\right) \ . \tag{5.52}$$

This is the electric analogue of the constraint relation (5.9) concerning the dressed magnetic charges of the black hole. This condition will be implemented in the effective potential $\mathcal{V}_{BM}$ eq(5.37) by using a Lagrange multiplier ξ. The auxiliary field ξ should be thought as the analogue of the Lagrange multiplier λ used in the black hole case; see also table (5.45). Combining eq(5.37) with eq(5.51), we end with the following generalized effective scalar potential for the black membrane,

$$\widetilde{\mathcal{V}}_{BM} = \left(W_a W^a + W_I W^I\right) + \xi\left(W_a W^a - W_I W^I - e^{+2\sigma}\mathrm{q}^2\right). \tag{5.53}$$

Notice that lowering and rising indices of $SO(4)$ and $SO(20)$ are done with the usual Kronecker metric, that is $W_a = W^a$ and $W_I = W^I$. Those of $SO(4,20)$ are done with the metric $\eta_{\Lambda\Sigma}$.

5.2.1 Black membrane attractor equations

The effective scalar potential of the 6D black membrane can be also put in the form

$$\widetilde{\mathcal{V}}_{BM} = (1+\xi)\, W_a W^a + (1-\xi)\, W_I W^I - \xi e^{+2\sigma}\mathrm{q}^2. \tag{5.54}$$

The variation of $\widetilde{\mathcal{V}}_{BM}$ with respect to ξ gives precisely the condition (5.51); while the variation with respect to W_a and W_I give constraint eqs on the field moduli,

$$\begin{array}{lll} \frac{\delta\widetilde{\mathcal{V}}_{BM}}{\delta W^a} & = & (1+\xi)\, W_a \quad , \\ \frac{\delta\widetilde{\mathcal{V}}_{BM}}{\delta W^I} & = & (1-\xi)\, W_I \quad , \\ \frac{\delta\widetilde{\mathcal{V}}_{BM}}{\delta \xi} & = & W_a W^a - W_I W^I - e^{+2\sigma}\mathrm{q}^2 \quad . \end{array} \tag{5.55}$$

The attractor eqs for the black membrane corresponds to the extremum (minimum) of this potential. These eqs read as follows

$$(1+\xi)\, W_a = 0 \quad , \tag{5.56}$$

$$(1-\xi)\, W_I = 0 \quad , \tag{5.57}$$

$$\left(W_a W^a - W_I W^I\right) - e^{+2\sigma}\mathrm{q}^2 = 0 \quad . \tag{5.58}$$

Like for the black hole, there are three solutions extremizing the effective scalar potential $\widetilde{\mathcal{V}}_{BM}$. These solutions, which are classified by the sign of the semi-norm of the electric charge vector q_Λ, are listed below:
(1) *first solution* ($\mathrm{q}^2 = 0$):

$$\begin{array}{lll} W_a & = & 0 \quad , \\ W_I & = & 0 \quad , \\ \xi & \neq & \pm 1 \quad . \end{array} \tag{5.59}$$

This is a singular solution.
(2) *second solution* ($\mathrm{q}^2 < 0$):

$$\begin{array}{lll} W_a & = & 0 \quad , \\ W_I & \neq & 0 \quad , \\ \xi & = & +1 \quad . \end{array} \tag{5.60}$$

This solution corresponds to *non BPS* black membrane.
(3) *third solution* ($\mathrm{q}^2 > 0$):

$$\begin{array}{lll} W_a & \neq & 0 \quad , \\ W_I & = & 0 \quad , \\ \xi & = & -1 \quad . \end{array} \tag{5.61}$$

This solution corresponds to *BPS* black membrane.
Putting these solutions back into (5.54), we can determine the value $\widetilde{\mathcal{V}}_{BM}^{\min}$ of the effective potential at the extremum. For the three solutions, the extremal values can be combined altogether in a unique form given by:

$$\widetilde{\mathcal{V}}_{BM}^{\min} = -\xi e^{+2\sigma} \mathrm{q}^2 \ . \tag{5.62}$$

Notice that due to the constraint eq(5.58) which requires $\mathrm{q}^2 = 0$ for $W_a = W_I = 0$, the potential at the first extremum (first solution) should vanish:

$$(1): \qquad \widetilde{\mathcal{V}}_{BM}^{\min} = 0 \ . \tag{5.63}$$

For the two other cases (2) and (3) with $\mathrm{q}^2 \neq 0$, the values of the effective potential at the corresponding extremum read as follows:

$$\widetilde{\mathcal{V}}_{BM}^{\min} = e^{+2\sigma} \left|\mathrm{q}^2\right| > 0 \ , \tag{5.64}$$

where the dependence into the Lagrange parameter ξ has been also fixed as $\xi = \pm 1$.
Notice that $\xi = +1$ corresponds to the non BPS black membrane while $\xi = 1$ is a BPS state.
Notice also that the value of the effective potential at the extremums depends on the factor $e^{+2\sigma}$ which, like in the case of the black hole, is an unfixed modulus.
Below, we give more details concerning the above solutions; in particular those solutions with

$$\widetilde{\mathcal{V}}_{BM}^{\min} > 0 \ . \tag{5.65}$$

Then, we turn to study the free factor $e^{\pm 2\sigma}$ and show how it can be fixed in the case of the dyonic attractor pair *BH-BM.*

5.2.2 Solving eq(5.55)

Recall that W^a and W^I depend, in addition to $\left(L^{-1}\right)^{\Lambda\Sigma}$, on the dilaton σ in the following manner eq(5.48),

$$\begin{array}{lll} W^a & = & e^{+\sigma} T^a \quad , \\ W^I & = & e^{+\sigma} T^I \quad , \end{array} \tag{5.66}$$

with

$$\begin{array}{lll} T^a & = & \mathrm{q}_\Lambda \left(L^{-1}\right)^{\Lambda a} \quad , \\ T^I & = & \mathrm{q}_\Lambda \left(L^{-1}\right)^{\Lambda I} \quad . \end{array} \tag{5.67}$$

Using this factorization, we will show that there are various ways to solve the attractor eqs of the black membrane.

Among these solutions, we have the degenerate one associated with $W_a = 0 = W_I$ and leading to

$$\widetilde{\mathcal{V}}_{BM}^{\min} = 0. \tag{5.68}$$

This solution will be ignored hereafter.
The two other solutions are those associated with eqs(5.60-5.61). We have:

A. case: $W_a = 0$, $W_I \neq 0$, $\xi = 1$.

Since the W's depends on the moduli and the bare charges; i.e,

$$W = W(\sigma, L, q) \ , \tag{5.69}$$

the conditions $W_a = 0$ and $W_I \neq 0$ allows then to give the relation between the field moduli of (5.4) and electric charges q_Λ of the black membrane.
Substituting W_a and W_I in terms of T_a and T_I, we have

$$\begin{array}{lllll} W_a & = & e^{+\sigma}T^a & = & 0 \quad , \\ W_I & = & e^{+\sigma}T^I & \neq & 0 \quad . \end{array} \tag{5.70}$$

Obviously, the solutions of the above relations should satisfy the constraint equation

$$W_aW^a - W_IW^I = e^{+2\sigma}\mathrm{q}^2 \ , \tag{5.71}$$

which, by substituting $W_a = 0$, reduces to

$$\sum_{I=1}^{20} W_IW^I = \sum_{I=1}^{20} W_I^2 = -e^{+2\sigma}\mathrm{q}^2 > 0 \ . \tag{5.72}$$

As we see, definite positivity of the norm $\sum_{I=1}^{20} W_I^2$ requires

$$\mathrm{q}^2 = -\left|\mathrm{q}^2\right| < 0 \ . \tag{5.73}$$

Eq(5.70) can be solved in two basic ways as follows:
(**a**) either by taking $\sigma \to -\infty$ whatever the values of T^a; in particular $T^a \neq 0$. But this solution should be ruled out since we should have

$$W_I = e^{+\sigma}T_I \neq 0 \ , \tag{5.74}$$

which violates eq(5.72).
(**b**) or by taking $\sigma = \sigma_2$, an arbitrary but *a finite* number (say $\sigma_2 < \infty$), and $T^a = 0$ but $T^I \neq 0$.
A solution for $T^a = 0$ depends of the value of $\mathrm{q}^2 = \mathrm{q}_\Lambda\eta^{\Lambda\Sigma}\mathrm{q}_\Sigma$ and can, a priori, be split into two situations (i) and (ii) corresponding respectively to:

(i) *a light like charge vector* $\mathrm{q}^2 = 0$

We already know that this case should be ruled out; but it is interesting to see the explicit relation between the field moduli $L_{\Lambda\Sigma}$ of (5.4) and the electric charges of the black membrane. We have

$$\begin{array}{llll} \left(L^{-1}\right)^{\Lambda a} & = & \# \ q^\Lambda q^a & , \\ T^a & = & \# \ \mathrm{q}^2 q^a \quad = 0 & . \end{array} \tag{5.75}$$

However, because of eq(5.72) which requires

$$\sum_{I=1}^{20} W_I W^I \quad = \quad e^{+2\sigma_2} \sum_{I=1}^{20} T_I T^I \quad = \quad -e^{+2\sigma_2} \mathrm{q}^2 \quad , \tag{5.76}$$

we get

$$\sum_{I=1}^{20} T_I T^I = 0 \qquad \Rightarrow \qquad T_I = 0 \ . \tag{5.77}$$

This solution should be then ruled out since $T_I \neq 0$.

(ii) *a non zero semi-norm* $\mathrm{q}^2 \neq 0$

We have the following:

$$\begin{array}{lcll} \left(L^{-1}\right)^{\Lambda a} & = & \frac{\left(q^\Lambda q^a - \mathrm{q}^2 \eta^{\Lambda a}\right)}{\mathrm{q}^2} & , \\ T_a & = & \frac{\left(\mathrm{q}^2 q^a - \mathrm{q}^2 q^a\right)}{\mathrm{q}^2} \quad = 0 & . \end{array} \tag{5.78}$$

This solution is acceptable provided $\mathrm{q}^2 < 0$ since eq(5.72) requires

$$\sum_{I=1}^{20} T_I T^I = -\ \mathrm{q}^2 > 0 \ . \tag{5.79}$$

From this relation we can determine T^I; i.e

$$T^I = \frac{q^I \sqrt{-\mathrm{q}^2}}{\left(\sum_{J=1}^{20} q_J^2\right)} \qquad , \qquad I = 1, ..., 20 \ , \tag{5.80}$$

which leads in turns to

$$\left(L^{-1}\right)^{\Lambda J} = \mathrm{q}^\Lambda \mathrm{q}^J \frac{\sqrt{-\mathrm{q}^2}}{\mathrm{q}^2 \left(\sum_{I=1}^{20} q_I^2\right)} \qquad , \qquad I = 1, ..., 20 \ . \tag{5.81}$$

In this case, the values of the effective scalar potential $\widetilde{\mathcal{V}}_{BM}$ at the minimum is given by

$$\widetilde{\mathcal{V}}_{BM}^{\min} = -e^{+2\sigma_2} \mathrm{q}^2 = e^{+2\sigma_2} \left|\mathrm{q}^2\right| > 0 \ . \tag{5.82}$$

It depends on the electric charges. but it has a free dependence in the value σ_2 of the dilaton.

B. case : $W_a \neq 0$, $W_I = 0$, $\xi = -1$

We have to solve

$$\begin{array}{lcllcl} W_a & = & e^{+\sigma} T^a & \neq & 0 & , \\ W_I & = & e^{+\sigma} T^I & = & 0 & , \\ \xi & = & -1 & & & , \end{array} \tag{5.83}$$

with the constraint relation

$$\sum_{a=1}^{4} W_a W^a \quad = \quad \sum_{a=1}^{4} W_a^2 \quad = \quad e^{+2\sigma} \mathrm{q}^2 > 0 \quad . \tag{5.84}$$

From this constraint relation we see that the electric charges of the black membrane should be $\mathrm{q}^2 > 0$.
The method is quite similar to the one used for the black hole case. After some straightforward calculations, we end with the following

$$\text{case (3)}: \quad \widetilde{\mathcal{V}}_{BM}^{\min} = e^{+2\sigma_2}\mathrm{q}^2 \quad > 0 \quad , \tag{5.85}$$

where we still have the unfixed factor $e^{+2\sigma_2}$.

6. Entropy of the pair *BH-BM*

We start by recalling the various expressions of the effective scalar potentials of the 6D black attractors that have been obtained so far. These are collected in the following table

$$\begin{array}{lcl} \underline{\textit{6D black attractors}} & : & \underline{\textit{effective scalar potential}} \\ & & \\ \text{dyonic black string} & : & \mathcal{V}_{BS}(\sigma) = \frac{\mathrm{q}_0^2}{2}e^{4\sigma} + \frac{\mathrm{g}_0^2}{2}e^{-4\sigma} \\ & & \\ \text{black hole} & : & \mathcal{V}_{BH}(\sigma, R, \lambda) = e^{-2\sigma}\mathcal{V}_0(R, \lambda) \\ & & \\ \text{black 2-brane} & : & \mathcal{V}_{BM}(\sigma, T, \xi) = e^{2\sigma}\mathcal{V}_2(T, \xi) \end{array} \tag{6.1}$$

The entropy $\mathcal{S}_{BS}$ of the dyonic *6D* black string *BS* reads, in terms of the electric q_0 and magnetic g_0 charges of the 3-form field strength $\mathcal{H}_3$, as follows:

$$\mathcal{S}_{BS} = \frac{|\mathrm{g}_0\mathrm{q}_0|}{4} \quad , \tag{6.2}$$

or again like

$$\mathcal{S}_{BS} = \frac{\pi}{2}|\mathrm{k}_0| \quad , \quad \mathrm{k}_0 \in \mathbb{Z}^* \quad . \tag{6.3}$$

For the entropies $\mathcal{S}_{BH}$ and $\mathcal{S}_{BM}$ of the *6D* black hole *BH* and black membrane *BM*, the situation is a little bit different.
Viewed separately, the corresponding entropies $\mathcal{S}_{BH}$ and $\mathcal{S}_{BM}$ are respectively given by:

$$\mathcal{S}_{BH} = \frac{1}{4}e^{-2\sigma_0}\left|\mathrm{g}^2\right| \quad , \tag{6.4}$$

and

$$\mathcal{S}_{BM} = \frac{1}{4}e^{2\sigma_2}\left|\mathrm{q}^2\right| \quad , \tag{6.5}$$

with

$$\begin{aligned} \mathrm{g}^2 &= \left(\sum_{a=1}^4 g_a^2 - \sum_{I=1}^{20} g_I^2\right) \quad , \\ \mathrm{q}^2 &= \left(\sum_{a=1}^4 q_a^2 - \sum_{I=1}^{20} q_I^2\right) \quad . \end{aligned} \tag{6.6}$$

In these relations σ_0 and σ_2 stand respectively for the value of the dilaton field at the horizon of the black hole *BH* and the black membrane *BM*:

$$\begin{array}{lcl} \sigma_0 = \sigma(r_{bh}) & , & r_{bh} = \text{ black hole horizon} \quad , \\ \sigma_2 = \sigma(r_{bm}) & , & r_{bm} = \text{ black 2-brane horizon} \quad . \end{array} \tag{6.7}$$

As we have noted before, σ_0 and σ_2 might also take finite values but unfortunately cannot be fixed if dealing with BH and BM as independent objects.
A way to see why the effective potentials

$$\mathcal{V}_{BH} = \mathcal{V}_{BH}\left(\sigma, R, \lambda\right), \tag{6.8}$$

and

$$\mathcal{V}_{BM} = \mathcal{V}_{BM}\left(\sigma, T, \xi\right), \tag{6.9}$$

cannot fix the dilaton at their extremum is to note the following:
(i) First the scalar potentials $\mathcal{V}_{BH}$ and $\mathcal{V}_{BM}$ are eigenfunctions of the operator $\frac{d}{d\sigma}$:

$$\begin{array}{lcll} \frac{d\mathcal{V}_{BH}}{d\sigma} & = & -2\mathcal{V}_{BH} & , \\ \frac{d\mathcal{V}_{BM}}{d\sigma} & = & +2\mathcal{V}_{BM} & . \end{array} \tag{6.10}$$

(ii) the zeros of the effective potentials $\mathcal{V}_{BH}$ and $\mathcal{V}_{BM}$ can be obtained in three ways.
In the case of the black hole *BH*, the zeros are given by,

$$\mathcal{V}_{BH} \quad = \quad e^{-2\sigma_0}\mathcal{V}_0\left(R, \lambda\right) \quad = 0 \quad \Rightarrow \quad \begin{cases} (1) : e^{-2\sigma_0} = 0, & \mathcal{V}_0\left(R, \lambda\right) = 0 \\ (2) : e^{-2\sigma_0} = 0, & \mathcal{V}_0\left(R, \lambda\right) \neq 0 \\ (3) : e^{-2\sigma_0} \neq 0, & \mathcal{V}_0\left(R, \lambda\right) = 0 \end{cases} \tag{6.11}$$

For the configurations (1) and (2), the value σ_0 of the dilaton at the critical point is

$$\sigma_0 \rightarrow +\infty \ . \tag{6.12}$$

They lead to the degenerate relations (1.3-2.20).
However, for the third configuration, the critical value of the dilaton is unfixed and can be any arbitrary; but finite, value. This is the case we are interested in here.
In the case of the black membrane *BM*, we have

$$\mathcal{V}_{BM} \quad = \quad e^{+2\sigma_2}\mathcal{V}_2\left(T, \xi\right) \quad = \quad 0 \quad \Rightarrow \quad \begin{cases} (1) : e^{+2\sigma_2} = 0, & \mathcal{V}_2\left(T, \xi\right) = 0 \\ (2) : e^{+2\sigma_2} = 0, & \mathcal{V}_2\left(T, \xi\right) \neq 0 \\ (3) : e^{+2\sigma_2} \neq 0, & \mathcal{V}_2\left(T, \xi\right) = 0 \end{cases} \tag{6.13}$$

The configurations (1) and (2) imply

$$\sigma_2 \rightarrow -\infty \ , \tag{6.14}$$

while for the third configuration leaves σ_2 an arbitrary finite number.
Notice that eq(6.11) and (6.13) exhibit very remarkable properties; in particular the two following:
(**a**) They are exchanged under electric/magnetic duality.
At the black hole and the black membrane horizons, we then have

$$\begin{array}{lcll} \pm\sigma_0 & \leftrightarrow & \mp\sigma_2 & , \\ g_\Lambda & \leftrightarrow & q_\Lambda & . \end{array} \tag{6.15}$$

(**b**) The above relation (6.15) should be associated with eq(4.3) of the dyonic black string. This property shows that

$$\sigma_2 = -\sigma_0 \ , \tag{6.16}$$

ending then with one unknown quantity; say σ_0, which remains unfixed.
Moreover, eq(6.15) teaches us that the black hole potential (5.20-5.21)

$$\mathcal{V}_{BH} = e^{-2\sigma}\mathcal{V}_0 \ , \tag{6.17}$$

and the black membrane potential (5.37-6.1)

$$\mathcal{V}_{BM} = e^{+2\sigma}\mathcal{V}_2 \ , \tag{6.18}$$

as two limits of the potential of the dyonic black pair *DP* $\equiv$ *BM-BH.*

$$\mathcal{V}_{DP} \simeq e^{-2\sigma}\mathcal{V}_0 + e^{+2\sigma}\mathcal{V}_2 \ . \tag{6.19}$$

In the limit $\sigma \to -\infty$, the potential $\mathcal{V}_{DP}$ of the dyonic pair reduces as

$$\mathcal{V}_{DP} \qquad \to \qquad \mathcal{V}_{BH} \ , \tag{6.20}$$

and in the limit $\sigma \to +\infty$, it behaves like,

$$\mathcal{V}_{DP} \qquad \to \qquad \mathcal{V}_{BM} \ . \tag{6.21}$$

To get the explicit expression of σ_0, we have to study the attractor mechanism of the dyonic attractor *DP* $\equiv$ *BM-BH.*

6.1 Attractor eqs for the dyonic *DP*

To begin, notice that the general moduli dependence of the effective scalar potential $\mathcal{V}_{DP}$ of the dyonic black attractor pair is given by,

$$\mathcal{V}_{DP} = \mathcal{V}\left(\sigma; R, T; \lambda, \xi, \zeta; q, g\right). \tag{6.22}$$

The set of parameters $\{\sigma, R, T, \lambda, \xi, \zeta, q, g\}$ is the general set of the possible moduli in which may depend the effective scalar potential and which are supposed to describe the attractor eqs of the *DP* dyonic pair. They are as follows:
(a) the R's and the T's are the dressed charges as in eqs(5.18-5.19) and (5.48-5.49);
(**b**) λ and ξ are the Lagrange multipliers given by eqs(5.15-5.16) and (5.53);
(c) $\mathrm{q} = (\mathrm{q}_\Lambda)$ and $\mathrm{g} = (\mathrm{g}_\Lambda)$ are the electric and magnetic charges given by eqs(3.31,3.32,3.33,3.34)
(d) the variable ζ is an extra Lagrange multiplier that will be described below.
Notice also that, like for the dyonic black string *BS*, the potential $\mathcal{V}_{DP}$ of the dyonic pair should be also invariant under the electric/magnetic duality (6.15).

Expression of $\mathcal{V}_{DP}$

The explicit expression of $\mathcal{V}_{DP}$ is given by the sum of:
(**i**) the effective scalar potential of the black hole (5.20-5.21),

$$\mathcal{V}_{BH} = \mathcal{V}_{BH}\left(\sigma, R, \lambda, g_\Lambda\right). \tag{6.23}$$

(**ii**) the effective scalar potential of the black membrane (5.37) which is dual to $\mathcal{V}_{BH}$,

$$\mathcal{V}_{BM} = \mathcal{V}_{BM}\left(\sigma, T, \xi, q_\Lambda\right) \ . \tag{6.24}$$

(**iii**) an extra term depending on the dressed electric and magnetic charges Z and W. This term is given by the constraint eq(5.42)

$$\mathcal{C} = \mathcal{C}\left(Z, W\right) \ , \tag{6.25}$$

capturing the electric/magnetic duality between the dressed charges of the black hole and the black membrane. It may be interpreted as the interaction term.
Then, we have

$$\mathcal{V}_{DP} = \mathcal{V}_{BH} \ + \mathcal{V}_{BM} + \zeta\mathcal{C} \ , \tag{6.26}$$

where ζ is a Lagrange multiplier used to implement the constraint (5.42) in the effective scalar potential of the DP dyonic pair.
In addition to the various electric and magnetic bare charges q_Λ and g_Λ, the dyonic potential $\mathcal{V}_{DP}$ depends on the *eighty one* field variables $(\sigma, L_{\Lambda\Sigma})$ of the moduli space; and on the three Lagrange multipliers λ, ξ and ζ.
While the dilaton appears in $\mathcal{V}_{DP}$ as $e^{\pm 2\sigma}$, the *eighty* field moduli $L_{\Lambda\Sigma}$ are involved in the game through the dressed charges,

$$\begin{array}{lll} R_a & = & R_a\left(L_{\Lambda\Sigma}\right) \quad , \\ R_I & = & R_I\left(L_{\Lambda\Sigma}\right) \quad , \end{array} \tag{6.27}$$

and

$$\begin{array}{lll} T_a & = & T_a\left(L_{\Lambda\Sigma}^{-1}\right) \quad , \\ T_I & = & T_I\left(L_{\Lambda\Sigma}^{-1}\right) \quad . \end{array} \tag{6.28}$$

By substituting $\mathcal{V}_{BH}$ and $\mathcal{V}_{BM}$ by their explicit expression, we can put the DP effective scalar potential $\mathcal{V}_{DP}$ in the form

$$\mathcal{V}_{DP} = e^{-2\sigma}\mathcal{V}_0 + e^{+2\sigma}\mathcal{V}_2 + \zeta\mathcal{C} \ , \tag{6.29}$$

where we have set

$$\mathcal{V}_0 \quad = \quad (1+\lambda)\textstyle\sum_a R_a R^a \quad + (1-\lambda)\sum_I R_I R^I \quad -\lambda \mathrm{g}^2 \quad , \tag{6.30}$$

and

$$\mathcal{V}_2 \quad = \quad (1+\xi)\textstyle\sum_a T_a T^a \quad + (1-\xi)\sum_I T_I T^I \quad -\xi \mathrm{q}^2 \quad , \tag{6.31}$$

as well as

$$\mathcal{C} = -\left(1 - \sum_{\Lambda,\Sigma=1}^{24} \eta^{\Lambda\Sigma} Z_\Lambda W_\Sigma\right) = -\left(1 - \sum_{\Lambda,\Sigma=1}^{24} \eta^{\Lambda\Sigma} R_\Lambda T_\Sigma\right) . \tag{6.32}$$

The equations defining the extremum (minimum) of the scalar potential $\mathcal{V}_{DP}$ are then given by the four following systems of eqs:

$$\begin{array}{lll} \frac{\delta\mathcal{V}_{DP}}{\delta R^a} & = & 0 \quad , \\ \frac{\delta\mathcal{V}_{DP}}{\delta R^I} & = & 0 \quad , \\ \frac{\delta\mathcal{V}_{DP}}{\delta\lambda} & = & 0 \quad , \end{array} \tag{6.33}$$

and

$$\begin{array}{lll}\frac{\delta\mathcal{V}_{DP}}{\delta T^a} & = & 0 \quad , \\ \frac{\delta\mathcal{V}_{DP}}{\delta T^I} & = & 0 \quad , \\ \frac{\delta\mathcal{V}_{DP}}{\delta\xi} & = & 0 \quad , \end{array} \tag{6.34}$$

as well as

$$\frac{\delta\mathcal{V}_{DP}}{\delta\zeta} = 0 \tag{6.35}$$

and finally

$$\frac{\delta\mathcal{V}_{DP}}{\delta\sigma} = 0. \tag{6.36}$$

Eqs(6.33) give relative extremums (minimums) associated with the black hole BH contribution.
Eqs(6.34) define relative extremums (minimums) associated with the black membrane BM.
Eq(6.35) captures the duality relation between the black hole BH and the black membrane BM.
Eq(6.36) is in some sense special; it gives the values of σ_0 and σ_2 we are after.
Below, we give the details on the solutions of these eqs.

6.2 Extremums of $\mathcal{V}_{DP}$

Here we study the extremums (minimums) of the potential (6.29). Since $\mathcal{V}_{DP}$ is multi-variables function, we shall proceed by steps in order to get these minimums:
(**1**) First we solve successively the eq(6.33), eq(6.34) and (6.35). These solutions fix the critical values of the field moduli and the Lagrange multipliers in terms of the electric and magnetic charges q_0, g_0, q_a, g_a and q_I, g_I;

$$\begin{array}{lll} R^{\min} & = & R\left(q,g\right) \quad , \\ T^{\min} & = & T\left(q,g\right) \quad , \\ \lambda^{\min} & = & \lambda\left(q,g\right) \quad , \\ \xi^{\min} & = & \xi\left(q,g\right) \quad , \\ \zeta^{\min} & = & \zeta\left(q,g\right) \quad . \end{array} \tag{6.37}$$

(**2**) Then, we substitute the obtained solutions back into eq(6.29) to get the new effective potential $\widetilde{\mathcal{V}}_{\text{DP}}$ namely

$$\widetilde{\mathcal{V}}_{\text{DP}} = e^{-2\sigma}\mathcal{V}_0^{\min} + e^{+2\sigma}\mathcal{V}_2^{\min} + \left(\zeta\mathcal{C}\right)^{\min}, \tag{6.38}$$

where now

$$\begin{array}{lll} \mathcal{V}_0^{\min} & = & \mathcal{V}_0\left(R^{\min}, T^{\min}, \lambda^{\min}, \zeta^{\min}\right) \quad , \\ \mathcal{V}_2^{\min} & = & \mathcal{V}_2\left(T^{\min}, R^{\min}, \xi^{\min}, \zeta^{\min}\right) \quad . \end{array} \tag{6.39}$$

(**3**) After that, we solve the attractor equation given by the minimization of (6.38), i.e

$$\frac{\delta\widetilde{\mathcal{V}}_{DP}}{\delta\sigma} = 0 \ , \tag{6.40}$$

in order to determine the critical values of σ at the extremums.

6.2.1 Solving eqs(6.33-6.34-6.35)

A) *solution of eqs(6.33)*:
By substituting eq(6.29) and (6.30), we can be put eqs(6.33) in the form

$$\begin{array}{lcll} (1+\lambda)\, R_a + \zeta T_a & = & 0 & , \\ (1-\lambda)\, R_I - \zeta T_I & = & 0 & , \\ R_a R^a - R_I R^I & = & \mathrm{g}^2 & . \end{array} \tag{6.41}$$

These equations have three types of solutions which can be classified according to whether the sign of g^2; that is $\mathrm{g}^2 = 0$, $\mathrm{g}^2 > 0$ or $\mathrm{g}^2 < 0$.

<u>*Case A1*</u> ($\mathrm{g}^2 = 0$):
In this case, the solution reads as:

$$\begin{array}{lcll} R_a^{\min} & = & 0 & , \\ R_I^{\min} & = & 0 & , \\ \lambda^{\min} & = & -1 & , \\ \zeta^{\min} & = & 0 & , \end{array} \tag{6.42}$$

and all remaining other moduli are free.

<u>*Case A2*</u> ($\mathrm{g}^2 > 0$):
Here the solution reads as:

$$\begin{array}{lcll} R_a^{\min} & = & \mathrm{g}_a \sqrt{|\mathrm{g}^2|} \left(\sum_{b=1}^{4} \mathrm{g}_b^2\right)^{-\frac{1}{2}} & , \\ (R_a R^a)^{\min} & = & \left|\mathrm{g}^2\right| & , \\ R_I^{\min} & = & 0, & , \\ \lambda^{\min} & = & -1 & , \\ \zeta^{\min} & = & 0 & , \end{array} \tag{6.43}$$

and all remaining other moduli are free.

<u>*Case A3*</u> ($\mathrm{g}^2 < 0$):
In this case the solution is given by:

$$\begin{array}{lcll} R_I^{\min} & = & \mathrm{g}_I \sqrt{|\mathrm{g}^2|} \left(\sum_{J=1}^{20} g_J^2\right)^{-\frac{1}{2}} & , \\ \left(R_I R^I\right)^{\min} & = & -\mathrm{g}^2 & , \\ R_a^{\min} & = & 0 & , \\ \lambda^{\min} & = & +1 & , \\ \zeta^{\min} & = & 0 & , \end{array} \tag{6.44}$$

and all remaining other moduli are free.
In all cases **A1**, **A2** and **A3** ($\mathrm{g}^2 = 0$, $\mathrm{g}^2 > 0$ and $\mathrm{g}^2 < 0$), we have

$$\mathcal{V}_0^{\min} = \left(R_I R^I\right)^{\min} = \left|\mathrm{g}^2\right| . \tag{6.45}$$

B) *Solution of* eqs(6.34):
Using eq(6.29) and (6.30), we can be put eqs(6.34) in the form

$$\begin{array}{lcll} (1+\xi)\, T_a + \zeta R_a & = & 0 & , \\ (1-\xi)\, T_I - \zeta R_I & = & 0 & , \\ T_a T^a - T_I T^I & = & \mathrm{q}^2 & . \end{array} \tag{6.46}$$

Here also we have three kinds of solutions depending on the signs of q^2. The solutions are quite similar to the previous cases; they are given by:

Case B1 ($\mathrm{q}^2 = 0$):
In this case the solution reads as:

$$\begin{array}{lcll} T_a^{\min} & = & 0 & , \\ T_I^{\min} & = & 0 & , \\ \xi^{\min} & = & -1 & , \\ \zeta^{\min} & = & 0 & , \end{array} \tag{6.47}$$

and all remaining moduli are free.

Case B2 ($\mathrm{q}^2 > 0$):
Here the solution is given by:

$$\begin{array}{lcll} T_a^{\min} & = & \mathrm{q}_a \sqrt{|\mathrm{q}^2|} \left(\sum_{b=1}^{4} q_b^2 \right)^{-\frac{1}{2}} & , \\ (T_a T^a)^{\min} & = & \mathrm{q}^2 & , \\ T_I^{\min} & = & 0 & , \\ \xi^{\min} & = & -1 & , \\ \zeta^{\min} & = & 0 & , \end{array} \tag{6.48}$$

and all remaining other moduli are free.

Case B3 ($\mathrm{q}^2 < 0$):
In this case the solution reads as:

$$\begin{array}{lcll} T_I^{\min} & = & \mathrm{q}_I \sqrt{-\mathrm{q}^2} \left(\sum_{J=1}^{20} q_J^2 \right)^{-\frac{1}{2}} & , \\ \left(T_I T^I\right)^{\min} & = & -\mathrm{q}^2 & , \\ T_a^{\min} & = & 0 & , \\ \xi^{\min} & = & +1 & , \\ \zeta^{\min} & = & 0 & , \end{array} \tag{6.49}$$

and all remaining moduli are free.
In all cases **B1**, **B2** and **B3** ($\mathrm{q}^2 = 0$, $\mathrm{q}^2 > 0$ and $\mathrm{q}^2 < 0$), we have

$$\mathcal{V}_2^{\min} = (T_a T^a)^{\min} = \left|\mathrm{q}^2\right| . \tag{6.50}$$

C. *solution of* eqs(6.35):
Eq(6.35) gives

$$T^a R_a - T^I R_I = \sum_{a=1}^{4} (T_a R_a) - \sum_{I=1}^{20} (T_I R_I) = k, \tag{6.51}$$

and is solved as:

*Case C1***:**
In this case, the solution corresponds to

$$\begin{array}{lcll} T^a R_a & = & k & , \\ T^I R_I & = & 0 & , \end{array} \tag{6.52}$$

and requires that $T^a R_a \neq 0$ and $T^a \neq 0$. Consistency with the solutions of eqs(6.33-6.34) implies:

$$\begin{array}{lll} T_a^{\min} & = & kR_a^{\min}\left(\sum_{b=1}^{4}\left(R_b R_b\right)^{\min}\right)^{-1} \quad , \\ T_I^{\min} & = & 0 \quad , \\ R_I^{\min} & = & 0 \quad . \end{array} \tag{6.53}$$

Moreover equating the expression of $T_a^{\min}$ given by case **B2** with the expression of $T_a^{\min}$ which we obtain by substituting $R_a^{\min}$ by its value given by case **A2**, we get the following identity

$$\left(\sum_{b=1}^{4}\left(T_b^{\min}T_b^{\min}\right)\right)\left(\sum_{b=1}^{4}R_b^{\min}R_b^{\min}\right) = k^2. \tag{6.54}$$

Using eq(6.48) and eq(6.43), we can obtain the following electric and magnetic duality relation

$$\mathrm{q}^2.\mathrm{g}^2 = k^2, \qquad \mathrm{q}^2 > 0, \qquad \mathrm{g}^2 > 0. \tag{6.55}$$

This electric/magnetic duality relation involves the squares of the vector charges q_Λ and g_Λ.

Case C2:

In this case, the solution is given by

$$\begin{array}{lll} T^a R_a & = & 0 \quad , \\ T^I R_I & = & -k \quad , \end{array} \tag{6.56}$$

and corresponds to:

$$\begin{array}{lll} T_I^{\min} & = & -kR_I^{\min}\left(\sum_{J=1}^{20}\left(R_J R_J\right)^{\min}\right)^{-1} \quad , \\ T_a^{\min} & = & 0 \quad , \\ R_a^{\min} & = & 0 \quad . \end{array} \tag{6.57}$$

Substituting the solution of $T_I^{\min}$ given by case **B3** and $R_I^{\min}$ given by case **A3**, we get the following identity

$$\left(\sum_{I=1}^{20}\left(T_I^{\min}T_I\right)^{\min}\right)\left(\sum_{I=1}^{20}\left(R_I^{\min}R_I\right)^{\min}\right) = k^2, \tag{6.58}$$

which corresponds to the electric/magnetic duality $\mathrm{q}^2.\mathrm{g}^2 = k^2$; but now with $\mathrm{q}^2 < 0$ and $\mathrm{g}^2 < 0$.

6.2.2 Solving eq(6.40)

Substituting $\mathcal{V}_0^{\min}$, $\mathcal{V}_2^{\min}$ and $(\zeta\mathcal{C})^{\min}$ by their expressions, we get the following effective scalar potential for the dilaton field

$$\widehat{\mathcal{V}}_{DP}(\sigma) = e^{-2\sigma}\left|\mathrm{g}^2\right| + e^{+2\sigma}\left|\mathrm{q}^2\right| \ . \tag{6.59}$$

This is as positive definite effective dyonic potential

$$\widehat{\mathcal{V}}_{DP}(\sigma) > 0 \ , \tag{6.60}$$

that depends, in addition to the dilaton σ, on the semi-norms q^2 and g^2 of the bare electric and magnetic charges of the dyonic pair *DP*.
Since the second derivative

$$\frac{d^2\widehat{\mathcal{V}}_{DP}}{d\sigma^2} \geq 0 \ , \tag{6.61}$$

the minimum of eq(6.59) is obtained by solving

$$\left(e^{+2\sigma}\left|\mathrm{q}^2\right| - e^{-2\sigma}\left|\mathrm{g}^2\right|\right) = 0. \tag{6.62}$$

The critical value σ_c of the dilaton solving this constraint relation is

$$\begin{array}{lll} e^{+2\sigma_c} & = & \sqrt{\frac{|\mathrm{g}^2|}{|\mathrm{q}^2|}} \qquad\qquad , \\ \sigma_c & = & \frac{1}{4}\left(\ln\left|\mathrm{g}^2\right| - \ln\left|\mathrm{q}^2\right|\right) \quad . \end{array} \tag{6.63}$$

Putting this solution back into eq(6.62), we get

$$\widehat{\mathcal{V}}^{\min}_{BH-BM} = 2\sqrt{|\mathrm{g}^2\mathrm{q}^2|} \ . \tag{6.64}$$

This relation should be compared with eqs(4.13-4.14).

In the end, we would like to add that the analysis given in this section extends directly to the the dyonic pairs

$$DP \equiv BH\text{-}B3B, \tag{6.65}$$

and

$$DP \equiv BS\text{-}BM, \tag{6.66}$$

of (3.13-3.14) of the *7D* $\mathcal{N}=2$ supergravity embedded in *11D* M-theory on K3.

7. Conclusion and discussion

In this paper, we have studied the extremal black brane attractors in the *6D* (resp. *7D*) $\mathcal{N} = 2$ supergravity limit of the *10D* type IIA superstring (resp. *11D* M-theory) on K3. In these limits, the classical entropy of electrically charge black branes *EBB* (resp. magnetically charged branes *MBB*) have *degenerate* values; see eqs(1.3,1.11,2.20,5.33).
In trying to understand this classical degeneracy, we have been lead to make a proposal where the degenerate value of the 6D black hole *BH* entropy

$$\mathcal{S}^{entropy}_{BH} \quad = \quad 0 \quad , \tag{7.1}$$

and the entropy of black membrane *BM*

$$\mathcal{S}^{entropy}_{BM} \quad = \quad 0 \quad , \tag{7.2}$$

appear as two singular limits of the classical entropy

$$\mathcal{S}^{entropy}_{BH-BM} \equiv \mathcal{S}^{entropy}_{DP} \tag{7.3}$$

of the bound state dual pair $(BH\text{-}BM)_{6D} \equiv DP$. This result applies as well for the *6D* dyonic black string $(BS)_{6D}$ and for the dual pair attractors $(BH\text{-}B3B)_{7D}$ and $(BS\text{-}BM)_{7D}$

of the $\mathcal{N}=2$ *7D* supergravity theory.
In analyzing the degeneracy of $\mathcal{S}_{EBB}^{entropy}=\mathcal{S}_{MBB}^{entropy}=0$, we have also found that electric/magnetic duality is a *universal symmetry* playing a central in the physics of *6D/7D* black attractors. Among our results, we mention the following:
First, by using the electric/magnetic duality (*e/m* symmetry for short), we have given a refined classification of the black attractors in *6D* and *7D*. These black branes are classified into two representations of the *e/m* symmetry: dyonic singlets and pairs as follows:

(1) *Six dimensions*

In *6D* non chiral supergravity theory with sixteen supercharges, we have:
(**a**) *An attractor singlet*, corresponding to the dyonic black string denoted as (*BS*). This dyonic attractor carries an electric charge $q_0=\left(\int_{S^3}^{*}\mathcal{F}_3\right)$ and a magnetic charge $g_0=\left(\int_{S^3}\mathcal{F}_3\right)$ with $\mathcal{F}_3$ being the field strength of the NS-NS $\mathcal{B}_{\mu\nu}$- field.
(**b**) *An attractor pair* describing the dual pair

$$DP\quad\equiv\quad BH\text{-}BM\quad\equiv\quad\begin{pmatrix}BH\\BM\end{pmatrix}\quad,\tag{7.4}$$

carrying *24* electric and *24* magnetic charges $\{q_\Lambda,g_\Lambda\}$.
The black hole *BH* carry *24* magnetic charge $g^\Lambda=\left(\int_{S^4}\mathcal{F}_4^\Lambda\right)$ and corresponds to the singular limit

$$\begin{array}{lll}\sigma & \rightarrow & +\infty\quad,\\ e^{-\sigma} & \rightarrow & 0\quad\quad,\end{array}\tag{7.5}$$

of the $SO(1,1)$ factor of the moduli space $SO(1,1)\times\frac{SO(4,20)}{SO(4)\times SO(20)}$. This singular limit may be formally stated as,

$$DP\quad\underrightarrow{\sigma\rightarrow+\infty}\quad\begin{pmatrix}BH\\0\end{pmatrix}\quad.\tag{7.6}$$

The same feature is valid for the electrically charged black membrane *BM* carrying *24* electric charges $\{q_\Lambda\}$. The *BM*, which is e/m dual to *BH*, corresponds to the singular limit

$$\begin{array}{lll}\sigma & \rightarrow & -\infty\quad,\\ e^{+\sigma} & \rightarrow & 0\quad\quad,\end{array}\tag{7.7}$$

in the moduli space. We also have

$$DP\quad\underrightarrow{\sigma\rightarrow-\infty}\quad\begin{pmatrix}0\\BM\end{pmatrix}\quad.\tag{7.8}$$

(2) *Seven dimensions*

In the 7D $\mathcal{N}=2$ supergravity theory we have no attractor singlet; but two pairs $(DP)_1$ and $(DP)_2$:
The first pair is given by the bound state *BH-B3B* carrying *22* electric and *22* magnetic charge $\{q_\Lambda,g_\Lambda\}$.

$$(DP)_1\quad\equiv\quad BH\text{-}B3B\quad\equiv\quad\begin{pmatrix}BH\\B3B\end{pmatrix}\quad,\tag{7.9}$$

The second attractor pair is given by the dual pair

$$(DP)_2 \quad \equiv \quad BS\text{-}BM \quad \equiv \quad \begin{pmatrix} BS \\ BM \end{pmatrix} \quad . \tag{7.10}$$

This pair carries an electric charge q_0 and a magnetic one g_0; it should compared with the 6D black string $(BS)_{6D}$.
Notice that the black hole $(BH)_{7D}$ (resp. *B3B*) with the *22* magnetic charges g_Λ (resp. *22* electric charges q_Λ) follows as the singular limit $\sigma \to +\infty$ (resp. $\sigma \to +\infty$) of $(DP)_1$.
The same property holds for the *7D* black string $(BS)_{7D}$ and the black membrane $(BM)_{7D}$. They are singular limits of the $(DP)_2$ pair.

Then we have considered the question of computing the entropies $\mathcal{S}_{black-brane}$ of the above *6D* and *7D* black attractors.
For the dyonic *6D* black string $(BS)_{6D}$ with an electric charge q_0 and a magnetic charge g_0, the entropy $\mathcal{S}_{BS}^{6D}$ is given by eqs(4.13-4.14) namely,

$$\mathcal{S}_{BS}^{6D} = \frac{1}{2}\mathrm{g}_0\mathrm{q}_0 > 0, \tag{7.11}$$

which, for later use, we prefer to rewrite as follows

$$\mathcal{S}_{BS}^{6D} = \frac{1}{2}\sqrt{\mathrm{g}_0^2\mathrm{q}_0^2} \, > 0. \tag{7.12}$$

Clearly $\mathcal{S}_{BS}^{6D}$ is invariant under e/m symmetry.
For the case of *6D* black hole $(BH)_{6D}$ and the *6D* black membrane $(BM)_{6D}$, the corresponding entropies $\mathcal{S}_{BH}^{6D}$ and $\mathcal{S}_{BM}^{6D}$ take degenerate values as in eqs(7.1-7.2).
Recall that this property of the classical entropy has been point out in literature many years ago [46]; see also [55, 56]. It is due to the specific structure of the scalar manifolds $\boldsymbol{M}_{6D}^{N=2}$ and $\boldsymbol{M}_{7D}^{N=2}$ of these the *6D* and *7D* theories which contain an ambiguous $SO(1,1)$ factor as shown below,

$$\begin{array}{lll} \boldsymbol{M}_{6D}^{N=2} & = & SO(1,1) \times \frac{SO(4,20)}{SO(4)\times SO(20)} \quad , \\ \boldsymbol{M}_{7D}^{N=2} & = & SO(1,1) \times \frac{SO(3,19)}{SO(3)\times SO(19)} \quad . \end{array} \tag{7.13}$$

The $SO(1,1)$ factor, which is associated with the dilaton, puts a very restrictive constraint on the critical value of the effective scalar potential and on the entropy.
Moreover, by freezing the dilaton to a some constant value; say $\sigma = \sigma_{BH}$ for the BH and $\sigma = \sigma_{BM}$ for the black membrane, the corresponding entropies are no longer zero; but they depend on these free constant parameters.
To overcome this difficulty, we have proposed that, classically, the black hole *BH* and the black membrane *BM* of the *6D* space time should be thought of as an attractor bound state with the singular limits (7.6,7.8). In this view, all the difficulties are overcome and *e/m* duality appears as a universal symmetry.

Entropy of dual pair DP

With the attractor bound state picture in mind, we have studied the attractor mechanism

of the *6D* dual pair $DP \equiv BH\text{-}BM$ and we have found, amongst others, the following:
(**i**) the values σ_{BH} and σ_{BM} of the dilaton at the horizons of $(BH)_{6D}$ and $(BM)_{7D}$ are as follows

$$\sigma_{BH} \quad = \quad -\sigma_{BM} \quad , \tag{7.14}$$

in agreement with e/m duality. Moreover we have been able to compute σ_{BH} which is given by

$$\sigma_{BH} \quad = \quad \tfrac{1}{4}\left(\ln \mathrm{g}^2\right) - \tfrac{1}{4}\left(\ln \mathrm{q}^2\right) \quad , \tag{7.15}$$

with

$$\begin{aligned} \mathrm{g}^2 \quad &= \quad \mathrm{g}^\Lambda \eta_{\Lambda\Sigma} \mathrm{g}^\Sigma \quad , \\ \mathrm{q}^2 \quad &= \quad \mathrm{q}^\Lambda \eta_{\Lambda\Sigma} \mathrm{q}^\Sigma \quad , \end{aligned} \tag{7.16}$$

where $\eta_{\Lambda\Sigma}$ is the metric of the tangent space $\mathbb{R}^{4,20}$.
(**ii**) the entropy of the six space-time dimension DP dual pair is given by

$$\mathcal{S}_{DP} = \frac{1}{2}\sqrt{|\mathrm{g}^2\mathrm{q}^2|} \, , \tag{7.17}$$

Notice that the relation (7.17) of the entropy $\mathcal{S}_{DP}$ is quite similar to the relation (7.12) giving the entropy of the *6D* black string.
At the end, we would like to add the two following:
First, the explicit analysis we have made for *6D* applies as well for the black pairs (7.9-7.10) in *7D* $\mathcal{N} = 2$ supergravity embedded in 11D M-theory on K3. The entropy of the attractor bounds $(DP)_1$ and $(DP)_1$ have similar expression as in eqs(7.17-7.16) with $\eta_{\Lambda\Sigma}$ being the metric of the flat space $\mathbb{R}^{3,19}$.
Second, the Lagrange multiplier method we have developed in section 5 seems to be the appropriate way to deal with the study of the critical points of the black branes effective potentials.

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8. Appendix: On effective potential in 6D and 7D

We begin by recalling that, with the exception of $D = 4$, $\mathcal{N} = 1$, 2 and $D = 5$, $\mathcal{N} = 2$, all supergravity theories contain scalar fields whose kinetic Lagrangian is described by σ-models of the form G/H. The symmetry group G is a non compact group acting as an

isometry group on the scalar manifold and H is the isotropy subgroup having the form $H = H_{\text{aut}} \otimes H_{\text{matter}}$. The subsymmetry H_{aut} is the automorphism group of the extended supersymmetric algebra and H_{matter} is related to the matter supermultiplets. For the list of the coset manifolds G/H and the automorphism groups of the various supergravity theories for any dimension D and number $\mathcal{N}$, see [60, 61]. For $D = 6$, $\mathcal{N} = 2$ and $D = 7$, $\mathcal{N} = 2$, these are given by eqs(2.2) and (2.3).
We aso recall that generic D supergravity theories with moduli space G/H have several specific properties shared by most of these theories. Amongst these features, we quote the three following:
(**1**) the group G acts linearly on the $(p+2)$- forms field strengths $\mathcal{F}_{a_1 \ldots a_{p+2}}$ corresponding to the various $(p+1)$- forms $\mathcal{A}_{a_1 \ldots a_{p+1}}$ appearing in the gravitational and matter multiplets.
(**2**) the properties of a given supergravity theory with fixed D and $\mathcal{N}$ are completely specified by the geometry of G/H, in particular in terms of the coset representatives $L = L_{\Lambda\Sigma}$ satisfying the gauge symmetry relation

$$L\left(\xi'\right) = gL\left(\xi\right) h\left(g, \xi\right) \quad , \quad g \in G, \;\; h \in H, \;\; \xi' = \xi'\left(\xi\right) \quad ,$$

with ξ being the coordinates of the coset G/H. In particular, the matrix $\mathcal{N}_{\Lambda\Sigma}$ capturing the field coupling metric of the $(p+2)$- forms $\mathcal{F}^{\Lambda}_{a_1 \ldots a_{p+2}}$ in the supergravity Lagrangian density is fixed in terms of L. The *physical* field strengths $\mathcal{T}^{\Lambda}_{a_1 \ldots a_{p+2}}$ of the *interacting* theories are also *dressed* with scalar fields as explicitly developed in the literature; especially in a series of papers by Ferrara and collaborators; see for instance sections 3 and 5 of the study [46] and [62]-[65] for a geometrical approach dealing with the so called $\widehat{\mathrm{F}}_4$ supergravity containing the *6D* $\mathcal{N} = 2$ superalgebra as a subsymmetry; see also the appendix B of [36].
(**3**) Like in 4D $\mathcal{N} = 2$ theory, higher dimensional supergravity exhibits as well two kinds of central charges: Z_{geo} coming from gravity multiplet (geometry) and Z_{matter} arising from the matter sector. The dressing property allows to write down the central charges $Z_{geo} \equiv Z_{a_1 \cdots a_p}$ associated to the $(p+1)$- forms $\mathcal{A}^{gravity}_{a_1 \ldots a_{p+1}}$ in *the gravitational multiplet* in terms of the geometrical structure of the moduli space. The matter $(p+1)$- forms $\mathcal{A}^{matter}_{a_1 \ldots a_{p+1}}$ of the matter multiplets give rise to charges that are closely related to the central charges. Notice in passing that when $p > 1$, the central charges do not appear in the usual supersymmetry algebra, but in its extended version containing the central generators $Z_{a_1 \cdots a_p}$ associated to p- dimensional extended objects. Notice also that besides the fact that they satisfy differential relations of Maurer- Cartan type, the central charges Z satisfy as well sum rules quite analogous to those for the $\mathcal{N} = 2$ special geometry case [66]. These sum rules, which define in particular the effective potential,

$$\mathcal{V}_{eff} \sim \left|Z_{geo}\right|^2 + \left|Z_{matter}\right|^2 . \tag{8.1}$$

have been analyzed in details in [46]. Our main goal below is to write down the explicit form of the dressed charges Z_{geo}, Z_{matter} in the 6D/7D supergravity cases and then $\mathcal{V}^{6D/7D}_{eff}$. We also give some useful relations between Z_{geo} and Z_{matter} which are analogous the familiar $D = 4$, $\mathcal{N} = 2$ supergravity using special geometry relations [66].
For concreetness, we shall first focus on $\mathcal{N} = 2$ supergravity in 6D and then move to 7D.

These theories have respectively *81* (*58*) scalars distributed as follows: **(i)** the dilaton σ belonging to the 6D (7D) $\mathcal{N}=2$ gravity multiplet. **(ii)** the eighty (fifty seven) other moduli ϕ_{aI} (ρ_{aI}) belonging to the 6D (7D) $\mathcal{N}=2$ Maxwell multiplets.

8.1 6D $\mathcal{N}=2$ supergravity

The effective scalar potential $\mathcal{V}_{eff}$ of the 6D black objects is given by the so called Weinhold potential (8.1) expressed as quadratics of the *dressed* charges [67, 68, 69, 70],

$$(Z_+, Z_-, Z_a, Z_I)\,, \qquad a=1,...,4, \qquad I=1,...,20. \tag{8.2}$$

These charges appear in the supersymmetric transformations of the (fermionic) fields of the 6D supergravity theory.
At the event horizon of the 6D black objects, the potential $\mathcal{V}_{eff}$ attains the minimum. The real (σ,ϕ_{aI}) moduli parameterizing $\frac{SO(1,1)\times SO(4,20)}{SO(4)\times SO(20)}$ are generally fixed by the charges

$$g^+,\quad g^-,\quad g^a,\quad h^I,\quad q_a,\quad p_I, \tag{8.3}$$

of the $\mathcal{N}=2$ 6D supergravity gauge field strengths

$$H_3^+ \quad,\quad H_3^- \quad,\quad F_2^a \quad,\quad F_2^I \quad,\quad F_{4a} \quad,\quad F_{4I}.$$

The attractor equations of the 6D $\mathcal{N}=2$ black objects are obtained from the minimization of the $(\mathcal{V}_{eff})_{\text{black}}$. Notice that from the field spectrum of the 6D $\mathcal{N}=2$ non chiral supergravity, one learns that two basic situations should be distinguished:
(**1**) 6D black string (BS) with near horizon geometry $AdS_3\times S^3$. This is a 6D dyonic black F- string solution. The electric/magnetic charges involved here are those of the gauge invariant 3- form field strengths

$$H_3^+ = \frac{1}{2}\left(H_3 + {}^\star H_3\right) \quad,\quad H_3^- = \frac{1}{2}\left(H_3 - {}^\star H_3\right),$$

associated with the usual 2- form antisymmetric $B_{\mu\nu}^{\pm}$ fields in 6D. The $\star$ conjugation stands for the usual Poincaré duality interchanging n- forms with $(6-n)$ ones.
(**2**) 6D black hole (BH) and its black 2- brane (BM) dual. The field strengths involved in these objects are related by the Poincaré duality in 6D space time which interchanges the 2- and 4- form field strengths.
Below, we study briefly and separately these two configurations.

Black string in 6D

The BPS black object of the 6D $\mathcal{N}=2$ non chiral theory is a dyonic string charged under both the self dual H_3^+ and anti-self dual H_3^- field strengths of the NS-NS B$^{\pm}$-fields. Using the following bare magnetic/electric charges,

$$g^{\pm} = \int_{S^3} H_3^{\pm}, \qquad g^{\pm} = \frac{1}{2}\left(g \pm e\right),$$

where $g=\int_{S^3} H_3$ and $e=\int_{S^3} {}^\star H_3$, one can write down the physical charges in terms of the dressed charges.

(a) *Dressed charges*

The dressed charges play an important role in the study of supergravity theories [46]. They appear in the supersymmetric transformations of the Fermi fields (here gravitinos), and generally read like

$$Z^{\pm} = X_{+}^{\pm} g^{+} + X_{-}^{\pm} g^{-}, \tag{8.4}$$

where the real 2×2 matrix

$$X = \begin{pmatrix} X_{+}^{+} & X_{-}^{+} \\ X_{-}^{+} & X_{+}^{+} \end{pmatrix},$$

parameterizes the $SO(1,1)$ factor of the moduli space $\widehat{G}$. Taking the η_{rs} flat metric as $\eta = diag(1,-1)$, we can express all the four real parameters $X_{-}^{\pm}$ and $X_{+}^{\pm}$ in terms of the dilaton $\sigma = \sigma(x)$ by solving the constraint eqs $X^{t}\eta X = \eta$ which split into four constraint relations like

$$\begin{array}{lll} X_{+}^{+}X_{+}^{+} - X_{+}^{-}X_{+}^{-} = 1 & , & X_{-}^{-}X_{-}^{-} - X_{-}^{+}X_{-}^{+} = 1 \quad , \\ X_{+}^{+}X_{-}^{+} - X_{+}^{-}X_{-}^{-} = 0 & , & X_{-}^{+}X_{+}^{+} - X_{-}^{-}X_{+}^{-} = 0 \quad . \end{array} \tag{8.5}$$

These eqs can be solved by,

$$X_{+}^{+} = X_{-}^{-} = \cosh(2\sigma), \qquad X_{+}^{-} = X_{-}^{+} = \sinh(2\sigma). \tag{8.6}$$

Putting these solutions back into the expressions of the central charges Z^{+} and Z^{-} (8.4), we get the following dilaton dependent quantities

$$Z^{\pm} = \frac{1}{2}\left[g \exp(-2\sigma) \pm e \exp(2\sigma)\right]. \tag{8.7}$$

Notice that these dressed charges have no dependence on the ω_{aI} field moduli of the coset $SO(4,20)/SO(4) \times SO(20)$. This is because the NS-NS B- fields is not charged under the isotropy group of the above coset manifold.

(b) *Black string potential*

With the dressed charges Z^{+} and Z^{-}, we can write down the gauge invariant effective scalar potential $\mathcal{V}_{BFS}$. It is given by the so called Weinhold potential,

$$\mathcal{V}_{BS} = \left(Z^{+}\right)^{2} + \left(Z^{-}\right)^{2}. \tag{8.8}$$

Notice that, as far symmetries are concerned, one also have the other *"orthogonal"* combination namely $\left(Z^{+}\right)^{2} - \left(Z^{-}\right)^{2}$. This combination corresponds just to the electric/magnetic charge quantization condition. By substituting eq(8.4) into the relation (8.8), we get the following form of the potential,

$$\mathcal{V}_{BS} = \left(g^{+}, g^{-}\right) \mathcal{M} \begin{pmatrix} g^{+} \\ g^{-} \end{pmatrix},$$

with

$$\mathcal{M} = \begin{pmatrix} \left(X_{+}^{+}\right)^{2} + \left(X_{+}^{-}\right)^{2} & 2X_{+}^{+}X_{-}^{+} \\ 2X_{+}^{-}X_{-}^{-} & \left(X_{-}^{-}\right)^{2} + \left(X_{-}^{+}\right)^{2} \end{pmatrix}.$$

From this matrix and using the transformations given in [68], we can read the gauge field coupling metric $\mathcal{N}_{+-}$ and $\mathcal{N}_{-+}$ that appear in the 6D $\mathcal{N}=2$ supergravity component field Lagrangian density

$$\frac{\mathcal{L}_{6D}^{N=2\text{ sugra}}}{\sqrt{-g}}=\mathcal{R}_6 1+\left(\frac{1}{2}\mathcal{N}_{+-}H^+\wedge H^-+\frac{1}{2}\mathcal{N}_{-+}H^-\wedge H^+\right)+\cdots$$

In this eq, $\mathcal{R}_6$ is the usual 6D scalar curvature and $g=\det\left(g_{\mu\nu}\right)$. By further using (8.7), we can put the potential $\mathcal{V}_{BFS}$ into the following form

$$\mathcal{V}_{BS}\left(\sigma\right)=\frac{g^2}{2}\exp\left(-4\sigma\right)+\frac{e^2}{2}\exp\left(4\sigma\right). \tag{8.9}$$

Notice that the self and anti- self duality properties of the field strengths H_3^+ and H_3^- imply that the corresponding magnetic/electric charges are related as $g^+=e^+$, $g^-=-e^-$. Using the quantization condition for the dyonic 6D black F- string namely $\left(e^+g^++g^-e^-\right)=2\pi k$, k integer, one gets,

$$\left(g^+g^+-g^-g^-\right)=eg=2\pi k. \tag{8.10}$$

Then the the quantity $\left(Z^+\right)^2-\left(Z^-\right)^2$ becomes $\left(Z^+\right)^2-\left(Z^-\right)^2=2eg$, being just the quantization condition of the electric/magentic charges of the F- string in 6D space time.

(2) *6D Black Hole*

Contrary to the dyonic BS, the 6D black hole is magnetically charged under the $U^4\left(1\right)\times U^{20}\left(1\right)$ gauge group symmetry generated by the gauge transformations of the $(4+20)$ gauge fields of the 6D $\mathcal{N}=2$ gravity fields spectrum. Recall that in 6D, the electric charges are given, in terms of the field strength F_{4a} and F_{4I}, by,

$$q_a=\int_{S^4}F_{4a}\quad,\quad p_I=\int_{S^4}F_{4I}\quad.$$

with $a=1,...,4$ and $.I=1,...,20$ The corresponding magnetic duals, which concern the black 2- brane, involve the 2- form field strengths F_2^Λ integrated over 2- sphere,

$$g^a=\int_{S^2}F_2^a\quad,\quad h^I=\int_{S^2}F_2^I\quad.$$

Like for black string, the charges $Q_\Lambda=\left(q_a,p_I\right)$ are not the physical ones. The physical charges; to be denoted like Z_a, Z_I, appear dressed by the 6D scalar fields parameterizing the moduli space of the 10D type IIA superstring on K3. Recall that the charges Z_a and Z_I appear respectively in the supersymmetric transformations of the *four* graviphotinos/dilatinos and the *twenty* photinos of the $\mathrm{U}^{20}\left(1\right)$ Maxwell multiplet of the gauge-matter sector.

(a) *Dressed charges*

The dressing of the *twenty four* electric charges $\left(q^a,p^I\right)$ of the gauge fields $\left(\mathcal{A}_\mu^a,\mathcal{A}_\mu^I\right)$ read as follows:

$$\begin{aligned}Z_a&=e^{-\sigma}\left(Y_{ab}q^b+\phi_{aJ}p^J\right),\\ Z_I&=e^{-\sigma}\left(V_{Ib}q^b+Y_{IJ}p^J\right).\end{aligned} \tag{8.11}$$

Using the real 24×24 matrix $M_{\Lambda\Sigma} = e^{-\sigma} \times L_{\Lambda\Sigma}$,

$$L_{\Lambda\Sigma} = \begin{pmatrix} Y_{ab} & \phi_{aJ} \\ V_{Ia} & Y_{IJ} \end{pmatrix},$$

that defines the moduli space $\widehat{G}$, the dressed charges $Z_\Lambda = (Z_a, Z_I)$ can be put in the condensed form

$$\begin{aligned} Z_a &= M_{a\Sigma} Q^\Sigma = e^{-\sigma} L_{a\Sigma} Q^\Sigma, \\ Z_I &= M_{I\Sigma} Q^\Sigma = e^{-\sigma} L_{I\Sigma} Q^\Sigma. \end{aligned} \tag{8.12}$$

Obviously not all the parameters carried by $L_{\Lambda\Sigma}$ are independent; the extra dependent degrees of freedom are fixed by imposing the $SO(4,20)$ orthogonality constraint eqs and requiring gauge invariance under $SO(4) \times SO(20)$. The factor $e^{-\sigma}$ of eq(8.11) is then associated with the non compact abelian factor $SO(1,1)$ considered previously.
Taking the $\eta_{\Lambda\Sigma}$ flat metric of the non compact group $SO(4,20)$ as $\eta_{\Lambda\Sigma} = diag\,(4\,(+)\,, 20\,(-))$, we can express all the $24 \times 24 = 576$ real parameters $L_{\Lambda\Sigma}$ in terms of eighty of them only; that is in terms of ϕ_{aJ}. Notice moreover that setting,

$$\begin{aligned} Z_a &= e^{-\sigma} R_a, \qquad R_a = \left(L_{a\Sigma} Q^\Sigma\right), \\ Z_I &= e^{-\sigma} R_I, \qquad R_I = \left(L_{I\Sigma} Q^\Sigma\right), \end{aligned} \tag{8.13}$$

as well as $L_\Sigma^\Upsilon \cdot E_{\digamma}^\Sigma = \left(L_a^\Upsilon E_{\digamma}^a - L_I^\Upsilon E_{\digamma}^I\right) = \delta_{\digamma}^\Upsilon$, one can compute a set of useful relations. In particular we have

$$\begin{aligned} dL_{\digamma\Lambda} &= L_{\Upsilon\Lambda} \cdot \left(dL_\Sigma^\Upsilon\right) \cdot P_{\digamma}^\Sigma, \\ \nabla Z_a &= \left(D^{H_1} Z_a + Z_a d\sigma\right), \\ \nabla Z_I &= \left(D^{H_2} Z_I + Z_I d\sigma\right), \end{aligned} \tag{8.14}$$

where

$$\begin{aligned} D^{H_1} Z_a &= \left(dZ_a - \Omega_a^b Z_b\right), \qquad H_1 = \mathcal{O}(4)\,, \\ D^{H_2} Z_I &= \left(dZ_I - \Omega_I^J Z_J\right), \qquad H_2 = \mathcal{O}(4)\,, \end{aligned} \tag{8.15}$$

and where Ω_a^b and P_a^I are given by

$$\Omega_a^b = E_a^\Sigma \cdot \left(dL_\Sigma^b\right), \qquad P_a^I = E_a^\Sigma \cdot \left(dL_\Sigma^I\right),$$

together with similar relation for Ω_I^J and P_I^a. Using (8.14), we can write down the Maurer-Cartan eqs for the dressed charge. They read as follows,

$$\nabla Z_a = P_a^I Z_I, \qquad \nabla Z_I = P_I^a Z_a. \tag{8.16}$$

Notice in passing that $Z_I = 0$ is a solution of $\nabla Z_a = 0$. The same property is valid for $Z_a = 0$ which solves $\nabla Z_I = 0$.

(b) *Effective black hole potential*
Using the dressed charges (8.11-8.12), we can write down the gauge invariant effective scalar potential $\mathcal{V}_{BH}$. Following [46, 69], $\mathcal{V}_{BH}$ reads as,

$$\mathcal{V}_{BH}(\sigma, L) = (Z_a Z^a) + \left(Z_I Z^I\right), \tag{8.17}$$

which can be also put in the form

$$\mathcal{V}_{BH}(\sigma, L) = e^{-2\sigma}\left[(R_a R^a) + \left(R_I R^I\right)\right].$$

Clearly $\mathcal{V}_{BH}$, which is positive, is manifestly gauge invariant under both:
(**a**) the $U^4(1) \times U^{20}(1)$ gauge transformations since the vectors Z_a and Z_I depend on the electric charges of the field strengths only which, as we know, are gauge invariant.
(**b**) the gauge transformations of the $SO(4) \times SO(20)$ isotropy group of the moduli space. $\mathcal{V}_{BH}$ is given by scalar products of the vectors Z_a and Z^a (resp Z^I and Z_I).
Using eqs(8.11), we can express the black hole potential as follows:

$$\mathcal{V}_{BH} = e^{-2\sigma}\left(q^a \mathcal{N}_{ab} q^b + q^a \mathcal{N}_{aJ} p^J + p^I \mathcal{N}_{Ib} q^b + p^I \mathcal{N}_{IJ} p^J\right),$$

or in a condensed manner like $\mathcal{V}_{BH} = e^{-2\sigma} Q^\Lambda \mathcal{N}_{\Lambda\Sigma} Q^\Sigma$ with

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \mathcal{N}_{ab} & \mathcal{N}_{aJ} \\ \mathcal{N}_{aJ} & \mathcal{N}_{IJ} \end{pmatrix}$$

Notice that, like for BS, $\mathcal{N}_{\Lambda\Sigma}$ has a 6D filed theoretical interpretation in terms of the gauge coupling of the gauge field strengths $\mathcal{F}^\Lambda_{\mu\nu}$; i.e a term like $\frac{1}{4}\sqrt{-g}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^{\mu\nu\Sigma}$ appears in the component fields of the 6D $\mathcal{N} = 2$ supergravity Lagrangian density.

8.2 7D N=2 supergravity

Here we discuss briefly the effective scalar potential of the black objects in 7D. This study is quite similar to the previous 6D analysis. Recall that the moduli space of this theory is given by $\frac{SO(3,19)\times SO(1,1)}{SO(3)\times SO(19)}$. In 7D space time, the bosonic fields content of the $\mathcal{N} = 2$ supergravity multiplet is given by

$$\left(g_{\mu\nu},\quad B_{[\mu\nu]},\quad \mathcal{A}^a_\mu,\quad \sigma\right), \qquad a = 1, 2, 3, \qquad \mu, \nu, \rho = 0, ..., 6,$$

where $B_{[\mu\nu]}$ is dual to a 3- form gauge field $C_{[\mu\nu\sigma]}$. There is also nineteen U(1) Maxwell with the following 6D bosons:

$$\left(\mathcal{A}^I_\mu \quad , \quad \rho^{aI}\right), \qquad a = 1, 2, 3, \qquad I = 1, ..., 19,$$

where ρ^{aI} capture 3×19 degrees of freedom. The gauge invariant $(p+2)$- forms of the 7D $\mathcal{N} = 2$ supergravity are given by

$$H_3 \sim dB_2, \qquad \mathcal{F}^a_2 \sim d\mathcal{A}^a, \qquad \mathcal{F}^I_2 \sim d\mathcal{A}^I.$$

Extending the above 6D study to the 7D case, one distinguishes:
(**i**) 7D black 2- brane (black membrane BM). The effective scalar potential of the BM is

$$\mathcal{V}_{BM}^{7D}(\sigma) \sim Z^2 = e^{-4\sigma} g^2,$$

with $g = \int_{S^3} H_3$. The extremum of this potential is given by $\sigma = \infty$. The value of the potential at the minimum is $\left[\mathcal{V}_{BM}^{7D}(\infty)\right]_{\min} = 0$ and so the entropy vanishes identically.
(**ii**) 7D black hole: The effective potential of this black hole is given by

$$\mathcal{V}_{BH}^{7D}(\sigma, L) = \sum_{a=1}^{3} Z_a Z^a + \sum_{I=1}^{19} Z_I Z^I, \tag{8.18}$$

where

$$Z_a = e^{-\sigma} L_{a\Lambda} g^{\Lambda}, \qquad Z_I = e^{-\sigma} L_{a\Lambda} g^{\Lambda}, \tag{8.19}$$

satisfying the constraint relation,

$$\sum_{a=1}^{3} Z_a Z^a - \sum_{I=1}^{19} Z_I Z^I = Q^2, \qquad \left(\sum_{a=1}^{3} q_a q^a - \sum_{I=1}^{19} p_I p^I\right) = Q^2$$

and $Q^{\Lambda} = \left(q^a, p^I\right)$ with $q^a = \int_{S^2} \mathcal{F}_2^a$, $p^I = \int_{S^2} \mathcal{F}_2^I$, $a = 1, 2, 3$, $I = 1, ..., 19$. The real 22×22 matrix

$$L_{a\Lambda} = \begin{pmatrix} L_{ab} & \rho_{aI} \\ V_{Ia} & L_{IJ} \end{pmatrix}, \tag{8.20}$$

is associated with the group manifold $SO(3, 19)/SO(3) \times SO(19)$. It is an orthogonal matrix satisfying $L^t \eta L = \eta$ with $\eta = diag\left[3(+), 19(-)\right]$. The $SO(3) \times SO(19)$ symmetry can be used to choose L_{ab} and L_{IJ} matrices as $L_{ab} - L_{ba} = 0$, $L_{IJ} - L_{JI} = 0$. Putting the relations (8.19) back into (8.18), we get $\mathcal{V}_{BH}^{7D}(\sigma, L) = e^{-2\sigma} Q^{\Lambda} \mathcal{N}_{\Lambda\Sigma} Q^{\Sigma}$ where $\mathcal{N}_{\Lambda\Sigma} = \left(L_{a\Lambda} L_{\Sigma}^a + L_{I\Lambda} L_{\Sigma}^I\right)$.

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